EXPONENTIALLY SMALL ESTIMATES FOR KAM THEOREM NEAR AN ELLIPTIC EQUILIBRIUM POINT

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Abstract. We give a precise statement of KAM theorem for a Hamiltonian system in a neighborhood of an elliptic equilibrium point. If the frequencies of the elliptic point satisfy a Diophantine condition, with exponent τ, and a nondegeneracy condition is fulfilled, we show that in a neighborhood of radius r the measure of the complement of the KAM tori is exponentially small in $(1/r)^{1/(\tau+1)}$. This result is obtained by putting the system in Birkhoff normal form up to an appropriate order, and the key point relies on giving accurate estimates for its terms.

1. Introduction

We consider a Hamiltonian, with n degrees of freedom, having the origin as an elliptic equilibrium point. In suitable canonical coordinates, this system can be put in the form

$$H(q,p)=\sum_{s\geq 2} H_s(q,p), \tag{1}$$

where H_s is a homogeneous polynomial of degree s in (q,p) for every $s\geq 2$, and

$$H_2(q,p)=\frac{1}{2}\sum_{j=1}^{n}\lambda_j\left(q_j^2+p_j^2\right). \tag{2}$$

We are concerned with the existence of *n-dimensional invariant tori* in a neighborhood of the elliptic point. We first see that the system (1–2) is nearly-integrable by putting it in *Birkhoff normal form* up to an appropriate degree $K\geq 4$, provided the frequency vector $\lambda=(\lambda_1,\ldots,\lambda_n)$ is nonresonant up to order K. A quantitative version of Birkhoff theorem, implicitly contained in [4], allows us to obtain estimates for the normal form. Like in [9], we consider action–angle variables in a neighborhood of radius r. Assuming a suitable nondegeneracy condition, we apply the known KAM theorem and show that most trajectories in a neighborhood of radius r lie in invariant tori: we get for the relative measure of their complement an estimate of the type $\mathcal{O}\left(r^{(K-3)/2}\right)$.

We assume that λ satisfies a Diophantine condition: with given $\tau>n-1$ and $\gamma>0$,

$$|k\cdot\lambda|\geq\frac{\gamma}{|k|^{\tau}}\qquad\forall k\in\mathbb{Z}^n\setminus\{0\}, \tag{3}$$

where we write $|k| = \sum_{j=1}^{n} |k_j|$. We say λ to be τ, γ*-Diophantine.* Our main contribution is to show that in this case the estimates for the Birkhoff normal form allow us to choose the degree K as a function of r, giving rise to an exponentially small estimate of the type

$$\exp\left\{-\left(\frac{1}{r}\right)^{1/(\tau+1)}\right\}$$

for the measure of the complement of the invariant set. We remark that the estimates given in [4] do not allow to obtain the exponent $1/(\tau+1)$, but a worse one. Nevertheless, we shall see that an improvement of that estimates leads to the announced exponent.

2. Estimates for the Birkhoff normal form

Given $K \geq 4$, assume that the frequency vector λ is nonresonant up to order K: $k \cdot \lambda \neq 0$ for $k \in \mathbb{Z}^n$, $0 < |k| \leq K$. The well-known *Birkhoff theorem* [1, 7] states that, in some neighborhood of the origin, there exists a canonical transformation $\Psi^{(K)}$, near to the identity map, such that $\mathcal{H}^{(K)} = H \circ \Psi^{(K)}$ is in Birkhoff normal form up to degree K:

$$\mathcal{H}^{(K)}(q,p) = \lambda \cdot I + \mathcal{Z}^{(K)}(I) + \mathcal{R}^{(K)}(q,p) = h^{(K)}(I) + \mathcal{R}^{(K)}(q,p), \tag{4}$$

with

$$\mathcal{Z}^{(K)}(I) = \sum_{\substack{4 \leq s \leq K \\ s \text{ even}}} \mathcal{Z}_s(I), \qquad \mathcal{R}^{(K)}(q,p) = \sum_{s \geq K+1} \mathcal{R}_s^{(K)}(q,p), \tag{5}$$

where every $\mathcal{Z}_s(I)$ (uniquely determined) is a homogeneous polynomial of degree $s/2$ in the action variables

$$I_j = \frac{1}{2}\left(q_j^2 + p_j^2\right), \qquad j = 1, \ldots, n,$$

and every $\mathcal{R}_s^{(K)}(q,p)$ is a homogeneous polynomial of degree s in (q,p). Since $h^{(K)}(I)$ is integrable, and in a neighborhood of radius r we have $\mathcal{R}^{(K)} = \mathcal{O}\left(r^{K+1}\right)$, it turns out that $\mathcal{H}^{(K)}$ is a nearly-integrable Hamiltonian near the origin. However, to apply KAM theorem to $\mathcal{H}^{(K)}$ we need quantitative estimates for its terms.

As in [4], we introduce the linear change to complex canonical coordinates

$$x_j = \frac{1}{\sqrt{2}}(q_j - ip_j), \qquad y_j = -\frac{i}{\sqrt{2}}(q_j + ip_j), \qquad j = 1, \ldots, n.$$

Note that q, p are real if $\overline{y} = ix$. We define $|(x,y)| := \max_{j=1,\ldots,n} \sqrt{|x_j|^2 + |y_j|^2}$. Given $r > 0$, the real and complex polydisks of radius r centered at the origin will be denoted $\mathcal{B}_r$ and $\widehat{\mathcal{B}}_r$, respectively.

For a given homogeneous polynomial $f_s(x,y) = \sum_{|l+m|=s} f_{l,m} x^l y^m$, we define the norm

$$\|f_s\| := \sum_{|l+m|=s} |f_{l,m}|$$

(we use the notation $x^l = x_1^{l_1} \cdots x_n^{l_n}$, $y^m = y_1^{m_1} \cdots y_n^{m_n}$).

Proposition 1 *Let* $H(x,y) = \sum_{s\ge 2} H_s$ *be a real Hamiltonian with* $H_2 = \lambda \cdot I$, *and assume that* $\|H_s\| \le c^{s-2}d$ *for* $s \ge 3$. *Given* $K \ge 4$, *assume that*

$$|k \cdot \lambda| \ge \alpha_K \qquad \forall k \in \mathbb{Z}^n,\ 0 < |k| \le K, \tag{6}$$

with $0 < \alpha_K \le 1$. *Then, there exists a real canonical transformation* $\Psi^{(K)}$, *near to the identity map, such that* $\mathcal{H}^{(K)} = H \circ \Psi^{(K)}$ *is in the Birkhoff normal form (4–5) up to degree* K. *With some constants* c_1, c_2, *one has:*

a) $\|\mathcal{Z}_s\| \le \dfrac{c_2 c_1^{s-2}(s-2)!}{\alpha_3 \cdots \alpha_{s-1}}$ *for* $4 \le s \le K$ *and* s *even.*

b) $\left\|\mathcal{R}_s^{(K)}\right\| \le \dfrac{c_2 c_1^{s-2}(K-3)!(K-2)^{s-K+1}}{\alpha_3 \cdots \alpha_{K-1}\alpha_K^{s-K+1}}$ *for* $s \ge K+1$.

c) *The transformation* $\Psi^{(K)}$ *is analytic on* $\widehat{\mathcal{B}}_{r_K^*}$, *where we define* $r_K^* := \dfrac{\alpha_K}{c_1 K}$.

These estimates rely on the results obtained in [4], although a direct application would give worse estimates, with α_K^{s-3} instead of $\alpha_3 \cdots \alpha_{s-1}$ in the denominators. The improvement comes from the fact that, in the construction of the normal form, the only small divisors which appear up to the obtainment of $\mathcal{Z}_s$ correspond to the orders $3, \ldots, s-1$. This is crucial in order to get the right exponent in the estimates given in the last section.

3. Applying KAM theorem

We first recall a usual statement of KAM theorem. Let us consider a nearly-integrable Hamiltonian written in action–angle variables

$$\mathcal{H}(\phi, I) = h(I) + f(\phi, I),$$

with $\phi \in \mathbb{T}^n$ and $I \in \mathcal{G} \subset \mathbb{R}^n$. To show that most of the trajectories of $\mathcal{H}$ lie in n-dimensional invariant tori, one usually imposes one of the following nondegeneracy conditions on the frequency map $\omega = \nabla h$:

$$\det\left(\frac{\partial \omega}{\partial I}(I)\right) \ne 0 \qquad \text{or} \qquad \det\begin{pmatrix} \frac{\partial \omega}{\partial I}(I) & \omega(I) \\ \omega(I)^\top & 0 \end{pmatrix} \ne 0$$

for every $I \in \mathcal{G}$. We call these conditions *Kolmogorov nondegeneracy* and *isoenergetic nondegeneracy,* respectively.

We denote $\mathcal{V}_\rho(\mathcal{G})$ a complex neighborhood of radius r around $\mathcal{G}$.

Theorem 2 (KAM theorem) *Consider the Hamiltonian* $\mathcal{H} = h(I) + f(\phi, I)$, *analytic for* $\phi \in \mathbb{T}^n$ *and* $I \in \mathcal{V}_\rho(\mathcal{G})$, *with* f *of size* ε. *Assume that* $\omega = \nabla h$ *is Kolmogorov or isoenergetically nondegenerate on* $\mathcal{G}$. *Let* $\gamma > 0$ *given. For some constants* C_1, C_2, C_3, *if*

$$\varepsilon \le C_1 \gamma^2, \qquad \gamma \le C_2 \rho,$$

then there exists $\mathcal{I} \subset \mathbb{T}^n \times \mathcal{G}$ *filled with* n*-dimensional invariant tori of* $\mathcal{H}$, *satisfying*

$$\operatorname{mes}\left[(\mathbb{T}^n \times \mathcal{G}) \setminus \mathcal{I}\right] \le C_3 (\operatorname{diam} \mathcal{G})^{n-1} \gamma. \tag{7}$$

For more detailed statements and proofs, see [8, 9, 2, 3, 5]. Given γ, it turns out that the invariant tori of $\mathcal{H}$ come from invariant tori of the unperturbed system h with frequencies satisfying a Diophantine condition of the type (3), with a fixed τ. Choosing $\gamma \sim \sqrt{\varepsilon}$, the measure of the complement in (7) becomes $\mathcal{O}(\sqrt{\varepsilon})$.

Now, our aim is to apply KAM theorem to the Hamiltonian $\mathcal{H}^{(K)} = h^{(K)} + \mathcal{R}^{(K)}$ introduced in (4–5). We put this Hamiltonian in action–angle variables by introducing the known canonical change

$$q_j = \sqrt{2I_j} \cdot \cos \phi_j, \qquad p_j = \sqrt{2I_j} \cdot \sin \phi_j, \qquad j = 1, \ldots, n.$$

However, KAM theorem cannot be applied in a direct way because the change to action–angle variables is not analytic at the hyperplanes $I_j = 0$. Like in [9], this fact forces us to remove a neighborhood of these hyperplanes. Thus, to obtain invariant tori in the neighborhood $\mathcal{B}_r$, we consider for the action variables the domain

$$\mathcal{G}_{r,\rho} := \left\{ I \in \mathbb{R}^n \,:\, I \geq 2\rho,\ |I|_\infty \leq \frac{r^2}{2} \right\},$$

with $\rho > 0$ (we use the notation $I \geq a$ to mean that $I_j \geq a$ for $j = 1, \ldots, n$). With a suitable choice of ρ, this reduction of the domain does not affect essentially the measure estimates given in the next proposition.

To apply KAM theorem to $\mathcal{H}^{(K)}$, we also have to require that the frequency map $\omega^{(K)} = \nabla h^{(K)}$ is Kolmogorov or isoenergetically nondegenerate. In fact we only assume the nondegeneracy at the origin itself, since this suffices to ensure it in a small neighborhood. The condition we impose involves the vector λ and the matrix

$$A := \frac{\partial^2 \mathcal{Z}_4}{\partial I^2} \,.$$

Nevertheless, we point out that higher order conditions are also possible.

Proposition 3 *In the same situation of proposition 1, assume also*

$$\det A \neq 0 \qquad \textit{or} \qquad \det \begin{pmatrix} A & \lambda \\ \lambda^\top & 0 \end{pmatrix} \neq 0. \tag{8}$$

Let r_K^* defined as in part (c) of proposition 1. For some constants c_3, c_4, if

$$0 < r \leq c_3 r_K^*, \tag{9}$$

then there exists $\mathcal{T}_r^{(K)} \subset \mathcal{B}_r$ filled with invariant tori of $\mathcal{H}^{(K)}$, satisfying

$$\operatorname{mes} \left[\mathcal{B}_r \setminus \mathcal{T}_r^{(K)} \right] \leq c_4 \left(\frac{7r}{r_K^*} \right)^{(K-3)/2} \cdot \operatorname{mes} \mathcal{B}_r \,. \tag{10}$$

This result is obtained applying KAM theorem on the domain $\mathcal{V}_\rho(\mathcal{G}_{r,\rho})$, with $\varepsilon \sim r^{K+1}$ and $\rho \sim r^{(K+1)/2}$. In fact, this is a more elaborated version of a result given in [9], where a measure estimate like (10) is obtained, also with the exponent $(K-3)/2$.

4. The Diophantine case

Finally, we assume that the frequency vector λ satisfies the Diophantine condition (3) with given τ and γ. We then take $\alpha_K = \frac{\gamma}{K^\tau}$ in (6) and hence condition (9) is fulfilled if we choose $K \sim (\gamma/r)^{1/(\tau+1)}$, leading to an exponentially small estimate.

Theorem 4 *Let $H(x,y) = \sum_{s\geq 2} H_s$ be a real Hamiltonian with $H_2 = \lambda \cdot I$, and assume $\|H_s\| \leq c^{s-2}d$ for $s \geq 3$. Assume that λ is τ, γ-Diophantine, with $\tau > n-1$ and $\gamma > 0$. Assume also one of the nondegeneracy conditions* (8)*. For some constants c_5, c_6, c_7, if*

$$0 < r \leq c_5 \gamma,$$

then there exists $T_r \subset \mathcal{B}_r$ filled with invariant tori of H, satisfying

$$\operatorname{mes}\,[\mathcal{B}_r \setminus T_r] \leq c_6 \exp\left\{ -\left(\frac{c_7\gamma}{r}\right)^{1/(\tau+1)} \right\} \cdot \operatorname{mes} \mathcal{B}_r \ .$$

A related result has been announced in [6] where, for a fixed KAM torus of a nearly-integrable Hamiltonian, it is shown that in a neighborhood of radius r there exist many invariant tori, and the measure of their complement is exponentially small in $1/r$.

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