POINCARÉ-MELNIKOV-ARNOLD METHOD FOR TWIST MAPS

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1. Introduction

A general theory for perturbations of an integrable planar map with a separatrix to a hyperbolic fixed point has been developed in a previous lecture [5]. The splitting of the perturbed invariant curves was measured, in first order with respect to the parameter of perturbation, by means of a periodic *Melnikov function* M defined on the unperturbed separatrix. In the case of planar twist maps, M has zero mean and therefore there exists a periodic function L (called the *Melnikov potential*) such that $M = L'$. Consequently, if L is not identically constant (respectively, has non-degenerate critical points), the separatrix splits (respectively, the perturbed curves cross transversely).

The aim of this lecture is to present a similar theory for more dimensions. The natural frame is to consider twist maps on cotangent bundles. Once the suitable definition of unperturbed separatrix has been introduced (a non-trivial problem in the high-dimensional case), a scalar function L can be defined on it, in such a way that L verifies the same properties than in the planar case. The derivation of L is easily related to variational principles, and the property of being a scalar function instead of a vectorial function like the classical Melnikov function, makes it more useful for computations and geometrical understanding. Even more, it allows the application of Morse theory to establish the minimal number of transverse homoclinic orbits.

The results to be presented in this lecture are valid for exact symplectic maps on arbitrary exact symplectic manifolds, that is, the twist character is not essential. We have restricted ourselves to twist maps only for simplicity. Full details of the ideas presented here are contained in [4], where another more general situation (i.e., the exact symplectic case) is studied. Related ideas can be found in [1, 15, 14, 8, 7].

2. The maps

A *twist map* F is a map from a connected subset U of the cotangent bundle of a manifold $\mathcal{M}$ (which can be non-compact) into U, which comes equipped with a twist generating function $\mathcal{L} : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ that satisfies

$$F^*(y\,\mathrm{d}x) - y\,\mathrm{d}x = Y\,\mathrm{d}X - y\,\mathrm{d}x = \mathrm{d}\mathcal{L}(x, X), \qquad (X, Y) = F(x, y),$$

where (x, y) are any cotangent coordinates on $T^*\mathcal{M}$, that is, x are coordinates on $\mathcal{M}$, extended to coordinates (x, y) in the obvious way. The symplectic form ω_0 on $T^*\mathcal{M}$ reads as $\omega_0 = \mathrm{d}x \wedge \mathrm{d}y$ in cotangent coordinates. This can also be written in a coordinate free manner. Given $\mathcal{L}$, one can retrieve the map (at least implicitly) from $y = -\partial_1 \mathcal{L}(x, X)$, and $Y = \partial_2 \mathcal{L}(x, X)$. This can be done globally (i.e., $U = T^*\mathcal{M}$) only when $\mathcal{M}$ is diffeomorphic to a fiber of $T^*\mathcal{M}$, for example when $\mathcal{M}$ is the covering space of $\mathbb{T}^n$ or a manifold of constant negative curvature.

Finally, let us denote by $\pi : T^*\mathcal{M} \to \mathcal{M}$ the canonical projection.

3. The theory

Assume now that we are given a smooth twist diffeomorphism F_0 on the cotangent bundle $T^*\mathcal{M}$. Let $\mathcal{L}_0$ be its twist generating function. We assume that there exists a *hyperbolic fixed point* z_0^∞ of F_0, such that its n-dimensional (unstable and stable) invariant manifolds $\mathcal{W}_0^{\mathrm{u,s}}$ are *doubled*, that is, they coincide: $\mathcal{W} := \mathcal{W}_0^{\mathrm{u}} = \mathcal{W}_0^{\mathrm{s}}$.

In the planar case, the *separatrix* consists of the intersection of the invariant curves, except for the hyperbolic fixed point, which is the only point where such invariant manifolds are not submanifolds of the cotangent bundle. In the high-dimensional case, the situation is more complicated. We can consider three topologies on the set $\mathcal{W}$: the one induced by the inclusion $\mathcal{W} \subset T^*\mathcal{M}$, and the two ones induced by the inclusions $\mathcal{W} \subset \mathcal{W}_0^{\mathrm{u,s}}$. We first define the *bifurcation set* Σ of this problem as the subset of $\mathcal{W}$ formed by the points such that the three topologies do not coincide. Then, the *separatrix* Λ is defined as its complementary in W, i.e.,

$$\Lambda := \mathcal{W} \setminus \Sigma.$$

With this definition, it turns out that Λ *is* a doubly asymptotic exact submanifold of $T^*\mathcal{M}$, invariant by F_0.

Next, consider a perturbed twist map F_ε, and let $\mathcal{L}_\varepsilon = \mathcal{L}_0 + \varepsilon \mathcal{L}_1 + \mathrm{O}(\varepsilon^2)$ be the twist generating function of F_ε. For $0 < |\varepsilon| \ll 1$, there exists a hyperbolic fixed point z_ε^∞ of F_ε, close to z_0^∞, and it is not restrictive to normalize the twist generating function by imposing $\mathcal{L}_\varepsilon(x_\varepsilon^\infty, x_\varepsilon^\infty) = 0$, where $x_\varepsilon^\infty = \pi(z_\varepsilon^\infty)$. In particular, $\mathcal{L}_1(x_0^\infty, x_0^\infty) = 0$, where $x_0^\infty = \pi(z_0^\infty)$. We now define the *Melnikov potential* in the same way as for planar twist maps [5]:

$$L : \Lambda \longrightarrow \mathbb{R}, \quad L(z) = \sum_{k \in \mathbb{Z}} \mathcal{L}_1(x_k, x_{k+1}), \qquad x_k = \pi(z_k),\ z_k = F_0^k(z),\ z \in \Lambda. \tag{1}$$

The Melnikov theory is based on the following properties of the Melnikov potential [4]:

- $L : \Lambda \to \mathbb{R}$ is well-defined, smooth and invariant under the action of the unperturbed map: $L \circ F_0 = L$. Consequently, L can be defined on the *reduced separatrix* $\Lambda^* = \Lambda / F_0$.
- The differential of the Melnikov potential $M = \mathrm{d}L$ (called the *Melnikov function*), measures, in first order in ε, the distance between the perturbed invariant manifolds, and is also defined on the reduced separatrix Λ^*.
- If $L \not\equiv$ constant, then the perturbed invariant manifolds $\mathcal{W}_\varepsilon^{\mathrm{u,s}}$ split for $0 < |\varepsilon| \ll 1$, i.e., they do not coincide.
- If L has a critical point at $z = z_0$ then, for $0 < |\varepsilon| \ll 1$, $\mathcal{W}_\varepsilon^{\mathrm{u,s}}$ intersect transversally on a homoclinic point near z_0.
- If the unperturbed invariant manifolds are *completely doubled* (i.e., $\Sigma = \{z_0^\infty\}$), the reduced separatrix is a *compact* n-dimensional manifold without boundary. Actually, if σ denotes the sign of the product of the eigenvalues with modulus greater that one of $DF_0(z_0^\infty)$, and $\mathbb{S}^{n-1}$ stands for the unit sphere of $\mathbb{R}^n$, then Λ^* is homeomorphic to

$\mathbb{S} \times \mathbb{S}^{n-1}$ for $\sigma = +$, and it is somewhat more complicated for $\sigma = -$ (for more details, see [4]).
In this situation, Morse theory applied to the Melnikov potential, thought as a function over Λ^*, gives the minimal number of transverse homoclinic orbits, under conditions of generic position.

- There exists a *variational principle*, in an analogous way to the one of the planar case [9, 6], which establishes that the homoclinic orbits of a twist map with twist generating function $\mathcal{L}$ are the extremals of the *homoclinic action*

$$W[\mathcal{O}] := \sum_{k\in\mathbb{Z}} \mathcal{L}(x_k, x_{k+1}), \qquad \mathcal{O} = (x_k)_{k\in\mathbb{Z}},$$

and a *homoclinic area* can be defined for every pair of homoclinic orbits $\mathcal{O} = (x_k)_{k\in\mathbb{Z}}$, $\mathcal{O}' = (x'_k)_{k\in\mathbb{Z}}$, and is given by the difference of homoclinic actions $\Delta W[\mathcal{O}, \mathcal{O}'] = W[\mathcal{O}'] - W[\mathcal{O}]$. In terms of the Melnikov potential, there is also a nice expression for the homoclinic area:

$$\Delta W[\mathcal{O}, \mathcal{O}'] = \varepsilon \left(L(z'_0) - L(z_0)\right) + \mathrm{O}(\varepsilon^2).$$

We finish this survey of results with two remarks about different, but related, settings.

1. Regarding Hamiltonian flows, let $H_\varepsilon : T^*\mathcal{M} \times \mathbb{R} \to \mathbb{R}$ be a time-periodic Hamiltonian of period T, and $F_\varepsilon = \Psi_\varepsilon^T$ under the conditions of this section, where $\Psi_\varepsilon^t(z)$ is the solution of the associated Hamiltonian equations with initial condition z at $t = 0$. If $H = H_0 + \varepsilon H_1 + \mathrm{O}(\varepsilon^2)$, one can see [4] that the Melnikov potential takes the form (already known to Poincaré)

$$L(z) = -\int_{\mathbb{R}} H_1(\Psi_0^t(z), t)\,\mathrm{d}t,$$

where H_1 is determined by imposing $H_1(\Psi_0^t(z_0^\infty), t) \equiv 0$, or simply $H_1(z_0^\infty, t) \equiv 0$, if H_0 is autonomous.
2. In relation with a non-symplectic setting, let us assume now that $\mathcal{M} = \mathbb{R}^n$, that is, $T^*\mathcal{M} = \mathbb{R}^{2n}$. In that case, using a different point of view [13, 2], one can assume that the unperturbed map $F_0 : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ possesses n independent first integrals $H_1, \ldots, H_n$ on the separatrix Λ (not necessarily in involution, since this concept requires a symplectic structure), and consider a perturbation $F = F_0 + \varepsilon F_1 + \mathrm{O}(\varepsilon^2)$ (not necessarily symplectic). Then, the Melnikov vectorial function $M : \Lambda \to \mathbb{R}^n$ can be written as (compare with the planar case in [5]):

$$M = (M_1, \ldots, M_n)^\top, \qquad M_j(z) = \sum_{k\in\mathbb{Z}} \langle \nabla H_j(z_{k+1}), F_1(z_k)\rangle, \qquad z_k = F_0^k(z),\ z \in \Lambda.$$

4. The example

Let us consider *central* standard-like maps on $\mathbb{R}^{2n} = T^*\mathbb{R}^n$, that is,

$$F_0(x, y) = (y, -x + \nabla V_0(y)) \quad \text{or} \quad \mathcal{L}_0(x, X) = -\langle x, X\rangle + V_0(X) \tag{2}$$

where $V_0(x) = V_{\mathrm{c}}(\|x\|^2)$ for some function $V_{\mathrm{c}} : [0, \infty) \to \mathbb{R}$. Then, the "angular momenta" $A_{ij}(x, y) = x_i y_j - x_j y_i$ are first integrals and the $(n+1)$-dimensional manifold in $\mathbb{R}^{2n}$ of zero angular momenta is $\mathcal{A}_0^{n+1} := \{(x, y) : A_{ij}(x, y) = 0\} = \{(qa, pa) : a \in \mathbb{S}^{n-1},\ (q, p) \in \mathbb{R}^2\}$.

We now introduce the *reduced map* in $\mathcal{A}_0^{n+1}$ of F, as the planar standard-like map $f:\mathbb{R}^2\to\mathbb{R}^2$ defined by $f(q,p)=(p,-q+2V_c'(p^2)p)$. We note that

$$f(q,p)=(Q,P)\Longleftrightarrow F(qa,pa)=(Qa,Pa),\qquad \forall(q,p)\in\mathbb{R}^2,\ a\in\mathbb{S}^{n-1},\tag{3}$$

so that the non-trivial dynamics on the separatrix is induced by the reduced map.

To take advantage of the results for planar twist maps in the lecture [5], we introduce now the McLachlan map [10] as the central standard-like map with potential $V_0(y)=\mu\ln(1+\|y\|^2)$, $\mu>1$. It has the expression

$$F_0(x,y)=\left(y,-x+\frac{2\mu y}{1+\|y\|^2}\right),\qquad \mu>1.\tag{4}$$

Its reduced map is nothing else but the McMillan map [11] whose separatrix $\Gamma=\Gamma^+$ has the following natural parameterization [3]:

$$\Gamma=\{z_0(t)=(q_0(t),p_0(t))\},\qquad q_0(t)=p_0(t-h),\qquad p_0(t)=\sinh h\,\mathrm{sech}\,t,$$

where $h>0$ is determined by the equation

$$\cosh h=\mu.$$

Now, it is easy to check that

1. The origin is a hyperbolic fixed point of F_0, and $\mathrm{Spec}[DF_0(0)]=\{\mathrm{e}^{\pm h}\}$.
2. The invariant manifolds of F_0 are completely doubled, and the separatrix is given by
$$\Lambda=\{(qa,pa):(q,p)\in\Gamma,\ a\in\mathbb{S}^{n-1}\}.$$
3. The function $z_0:\mathbb{R}\times\mathbb{S}^{n-1}\longrightarrow\Lambda$ given by
$$z_0(t,a)=(p_0(t-h)a,p_0(t)a),\qquad p_0(t)=\sinh h\,\mathrm{sech}\,t,\tag{5}$$
is a *natural parameterization* of the separatrix, i.e., z_0 is a diffeomorphism that satisfies $F_0(z_0(t,a))=z_0(t+h,a)$, for $t\in\mathbb{R}$ and $a\in\mathbb{S}^{n-1}$.

As expected, we now consider a general perturbation of (4) that preserves the standard character, i.e.,

$$F_\varepsilon(x,y)=\left(y,-x+\frac{2\mu y}{1+\|y\|^2}+\varepsilon\nabla V(y)\right),\quad \mu>1,\ \varepsilon\in\mathbb{R},\tag{6}$$

with $V:\mathbb{R}^n\to\mathbb{R}$ determined by imposing $V(0)=0$. The generating function of F_ε that vanishes at the origin is $\mathcal{L}_\varepsilon=\mathcal{L}_0+\varepsilon\mathcal{L}_1$, where $\mathcal{L}_0(x,X)=-\langle x,X\rangle+\mu\ln(1+\|X\|^2)$ and $\mathcal{L}_1(x,X)=V(X)$. The Melnikov potential is simply

$$L:\mathbb{R}\times\mathbb{S}^{n-1}\to\mathbb{R},\qquad L(t,a)=\sum_{k\in\mathbb{Z}}V(p_0(t+hk)a),\qquad p_0(t)=\frac{\sinh h}{\cosh t}.\tag{7}$$

Since L is h-periodic in t (this is the invariance of the Melnikov potential under the action of the unperturbed map), we can consider t defined modulo h, i.e., L defined over the reduced separatrix $\mathbb{S}^1\times\mathbb{S}^{n-1}$.

Repeating the arguments for the case of the planar twist maps [5], we see that if V is a non-constant real entire function, then $V(p_0(t)a)$ has the same isolated singularities in the complex variable t as $p_0(t)$, and it is not difficult to check that they remain as singularities for the Melnikov potential, which must be non-constant. In this way we have established the following result.

Theorem 1 *If V is a non-constant real entire function, then the perturbed invariant manifolds of the standard-like map (6) split, for $0 < |\varepsilon| \ll 1$.*

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