Effective Computations in Celestial Mechanics and Astrodynamics *

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Abstract

Problems in Celestial Mechanics and Astrodynamics are considered under the point of view of Hamiltonian Dynamical Systems. A main tool to analyze the dynamics consists in studying the *skeleton* of the system, that is, the invariant objects (fixed points, periodic orbits and invariant tori) as well as their related stable, unstable and centre manifolds. Methods to compute these objects are presented. They are analytical, with the symbolic implementacion, purely numerical or a combination of both. Several examples are presented.

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1 Introduction

First we shall introduce some elementary ideas which are useful for the study of simple mathematical models of any physical problem. The reader familiar with these ideas can skip this section and proceed to the next one. For convenience, some definitions are given in both sections, in a slightly different language, making easier to skip this section.

1.1 Preliminaries on differential equations

Assume the problem to be studied is characterized by n variables, $v_1, v_2, ..., v_n$, or, shortly, v. To describe the evolution of v with respect to the time, we shall have a relation as

$$\dot{v} = f(t, v), \tag{1}$$

where $\dot{v}$ denotes the derivative of v with respect to the time t, or, in other words, the velocity of v as a function of v itself and time. The equation (1) is, in that case, the *nature law* that we assume describes the behaviour of the studied phenomenon. The equation (1) is an *ordinary differential equation.* If for a given time, $t^{(0)}$, the value, $v^{(0)}$, of the v variables is known, then the value of v, as a function of time, remains determined as far as (1) has a meaning. More concretely, we remember the

Theorem. *(Picard) Consider the following initial value problem: To determine a function $\varphi(t; t^{(0)}, v^{(0)})$ solution of (1) (that is, such that $\frac{d}{dt}\varphi(t; t^{(0)}, v^{(0)}) = f(t, \varphi(t; t^{(0)}, v^{(0)}))$ is satisfied for all t in a suitable interval) and such that the initial conditions hold: $\varphi(t^{(0)}; t^{(0)}, v^{(0)}) = v^{(0)}$. Then, if f is a continuous bounded function and it is locally Lipschitz with respect to v, such a function φ exists and is unique.*

An equation like (1) depends explicitely on time. When this is not the case ($\dot{v} = f(v)$) the equation is said to be autonomous. We can always reduce an equation to autonomous by adding an additional variable. Indeed, consider w with $n+1$ components $(w_0, w_1, \ldots, w_n)$ and take $w_0 = t, w_j = v_j$, if $j > 0$. Then

$$\begin{aligned} \dot{w}_0 &= 1, \\ \dot{w}_j &= f_j(w_0, w_1, ..., w_n), \;\; j = 1, ..., n, \end{aligned}$$

where f_j is the j–th component of f, that is, we have

$$\dot{w} = g(w), \tag{2}$$

which is autonomous. For autonomous equations the initial value of t, $t^{(0)}$, is irrelevant and can be taken as 0. Then the solution is simply written as $\varphi(t; x^{(0)})$ or $\varphi_t(x^{(0)})$ and is denoted as flow. If the solutions are defined, the following holds $\varphi_t(\varphi_s(x^{(0)})) = \varphi_{t+s}(x^{(0)})$ or, simply, $\varphi_t\varphi_s = \varphi_{t+s}$.

Let us consider how the solution of (2) depends on the initial conditions. If the solutions that for $t = 0$ start at $x^{(0)}$ and $y^{(0)}$ are defined for a given t, then

$$|\varphi(t, x^{(0)}) - \varphi(t, y^{(0)})| \leq e^{L|t-t_0|}|x^{(0)} - y^{(0)}|, \tag{3}$$

where L denotes the Lipschitz constant with respect to v , an inequality which follows from *Gronwall's Lemma.* In some cases the bound given by (3) is quite pessimistic. We would like to have something better and, furthermore, to know the behaviour of the flow, $\varphi_t(x^{(0)})$ with respect to $x^{(0)}$ when in (2) the function g is r times differentiable and $D^r g$ is continuous (that is, $g \in \mathcal{C}^r$). One has the result:

Theorem. *If g is $\mathcal{C}^r$ then φ_t is a diffeomorphism of class $\mathcal{C}^r$ (that is, φ_t is a 1 to 1 map and both φ_t and φ_t^{-1} are $\mathcal{C}^r$). Furthermore, the first differential, $D_x\varphi_t(x)$, satisfies the equation*

$$\frac{d}{dt}(D_x\varphi_t(x)) = Dg(\varphi_t(x))D_x\varphi_t(x) \tag{4}$$

with the initial condition $D_x\varphi_t(x)|_{t=0} = Id$, where Id denotes the identity matrix in $\mathbb{R}^n$.

Equation (4) is called first variational equation. Then

$$\varphi_t(y) - \varphi_t(x) = D_x\varphi_t(x)(y - x) + o(|y - x|),$$

which, quite often, gives bounds much better than (3) for $|y - x|$ small. Note that (4) is linear with time depending coefficients.

1.2 Continuous and discrete systems

If φ_t exists for all t real, then the property $\varphi_t\varphi_s = \varphi_{t+s}$ tells us that, if g is at least $\mathcal{C}^1$, the set of maps $\{\varphi_t, t \in \mathbb{R}\}$ is a one parameter group of diffeomorphisms. This suggest the following definition:

A *dynamical system* is a couple (M, φ_t), where M is a manifold (for instance, a surface, an open set of $\mathbb{R}^n$, etc) and $\{\varphi_t\}$ is a one parameter group of diffeomorphisms acting on M and satisfying $\varphi_t\varphi_s = \varphi_{t\circ s}$, where $t \circ s$ denotes the group operation. If, furthermore, μ is a measure in M and φ_t preserves μ (that is, $\mu(\varphi_t A) = \mu(A)$ for all measurable sets $A \in M$), then the triple (M, μ, φ_t) is called a *measure preserving dynamical system.*

Quite often the uniparametric groups are $\mathbb{R}$ or $\mathbb{Z}$. In the first case we shall talk about a *continuous dynamical system* and a differential equation like (2) can be associated to it, by defining $g(x) = \frac{d}{dt}\varphi_t(x)|_{t=0}$. If the group is $\mathbb{Z}$ we talk about a *discrete system* and it is only necessary to know φ_1, because the group property gives $\varphi_n = \varphi_1^n$ for all $n \in \mathbb{Z}$. The map φ_1 is denoted the generator of the group.

Of course, it is also possible to define dynamical systems (either continuous or discrete) when φ_t is simply a group of homeomorphisms, without any differentiability condition. If (M,μ) is a couple where M is an abstract space and μ a measure on it, one can also define an (abstract) dynamical system (M,μ,φ_t), where now φ_t are 1 to 1 maps, except by zero measure sets, preserving μ: $\varphi_t^{-1}(A)$ and A have the same measure for all measurable A.

If the maps φ_t have no inverse then we talk about *dynamical semisystems* (either continuous or discrete). If the semigroup is $\mathbb{R}_+$ (that is, the non negative reals), we shall still have $\varphi_t\varphi_s = \varphi_{t+s}$ if $t, s \geq 0$, but, in general, $\varphi_{-t} = \varphi_t^{-1}$ do not exists for $t > 0$. In what follows we shall be interested only in the case of systems.

It is quite simple to go from a continuous to a discrete system. Let T a fixed time. Then we define, as generator, ϕ_1, of the discrete group, the map $\varphi_{t=T}$ of the continuous system. If we start with an equation as (1) and f is $T-$periodic with respect to t ($f(t+T,x) = f(t,x)$ for all $t \in \mathbb{R}$ and allt x, provided f is defined), then there is no need to increase first the dimension of the space from n to $n+1$, as we did before to transform an equation like (1) to the form (2). Indeed, going to dimension $n+1$, the flow φ_T produces values of t which are coincident modulus the period. Hence, there is no need to give the "time" component of the w variables. It is true that, in this case, φ_T depends on the initial value of the time, $t^{(0)}$, but the maps produced by different initial times, as $t^{(0)}$ and $t^{(1)}$, are conjugated by the relation $\varphi(t^{(0)}+T;t^{(0)},\cdot)\,\varphi(t^{(0)};t^{(1)},\cdot) = \varphi(t^{(0)}+T;t^{(1)}+T,\cdot)\,\varphi(t^{(1)}+T;t^{(1)},\cdot)$, because, thanks to the periodicity one has $\varphi(t^{(0)}+T;t^{(1)}+T,\cdot) = \varphi(t^{(0)};t^{(1)},\cdot)$.

Another procedure to define a discrete system from a continuous one is the so called *Poincaré map* (or *return map*) when the initial differential equation is autonomous. Let $\mathcal{S}$ be an hypersurface in M (that is, of dimension one less that the one of M) transversal to the flow. This means that for every point $p \in \mathcal{S}$ the vector field, $g(p)$, is not contained in the tangent plane to $\mathcal{S}$ at p. Assume it exists a time, $\tau(p) > 0$, such that $\varphi_{\tau(p)} \in \mathcal{S}$. Then, by the implicit function theorem it exists a map defined in a neighbourhood V of p in $\mathcal{S}$, $P : V \to P(V)$ by $P(q) = \varphi_{\tau(q)}(q), \forall q \in V$. Of course, it can happen that $P(q) \notin V$ and then the new image $P(P(q))$ will not be defined. This can happen because, in general, the Poincaré sections, $\mathcal{S}$, are not "global". On the other hand, if we consider $t < 0$ there is no guarantee that for some $r \in V$, the set $\{\varphi_t(r), t < 0\}$ intersects $\mathcal{S}$. If it happens that, for q in a neighbourhood, perhaps smaller than V, V_0, we have $P^j(q)$ defined for all $j \in \mathbb{Z}$, then we have obtained a discrete system, with generator P, from the continuous one $\{\varphi_t\}$. Note that, in this case, the dimension of the space to which the variables belong (or *phase space*) has decreased by one unit. Obviously, if the initial differential equation, $\dot{x} = g(x)$, has several independent *first integrals* $F_1, F_2, ..., F_k$ (that is, functions such that $(\nabla F_j(x), g(x)) = 0$ with linearly independent gradients) a further reduction of the dimension is possible. If $p \in \mathcal{S}$ and $F_j(p) = c_j$ then the map P can be restricted to $V_0 \cap F_1^{-1}(c_1) \cap \ldots \cap F_k^{-1}(c_k)$, decreasing the dimension by k addicional units. This is the case of the Hamiltonian systems, which have the Hamiltonian itself, H, as first integral.

Reciprocally, given a discrete system of generator ϕ it is possible to definer a continuous system having ϕ as Poincaré map or as $\varphi_{t=1}$ map. The procedure is denoted as *suspension*. But one has to increase the dimension of the phase space by one unit or to introduce a non autonomous differencial equation (but 1-periodic in t). Furthermore, the new manifold containing the flow can be very complicated. It is possible to obtain in explicit form, at least if the discrete system is close to the identity, simple 1-periodic differentiable vector fields (and even $\mathcal{C}^\infty$ if the map is regular enough), but it is not so easy to have them analytic. We shall not use this technic in what follows, despite it is quite useful conceptually and for many proofs.

1.3 Invariant objects. Hyperbolicity

We shall use indistinctly flows or diffeomorphisms. To simplify the presentation we shall assume that we work on an open set of $\mathbb{R}^n$. First we look for "simple" solutions of (2). The simplests ones are the fixed points. A point, w^*, is called *fixed point* of (2) if $g(w^*) = 0$. Hence $\varphi_t(w^*) = w^*$ $\forall t \in \mathbb{R}$. The analysis of the behaviour close to a fixed point can be done by using (4), which, in the present case, has constant coefficients:

$$\frac{d}{dt}(D_x\varphi_t(x)) = Dg(w^*)D_x\varphi_t(x) \tag{5}$$

It is clear that, as this is a linear approximation, it will be necessary to take into account the terms of higher order for a full discussion. However, as we shall see, there are cases in which the information given by the linear part is enough to describe the behaviour close to w^*. The analysis of (5) is carried out by putting $Dg(w^*)$ in canonical form (Jordan form, for instance).

To present the cases in which the linear part gives enough information, a definition is required. We shall say that a matrix is *hyperbolic* (in the context of the differential equations) if it has all the eigenvalues with real part different from zero. In the case of dimension 2, if $Dg(w^*)$ is regular, only the case of *centers* is excluded. A *fixed point*, w^*, of (2) es called *hyperbolic* if $Dg(w^*)$ is an hyperbolic matrix (we assume $g \in \mathcal{C}^1$, of course). A basic result is given by

Theorem. *(Hartman) Let w^* be an hyperbolic fixed point of a differential equation $\dot{w} = g(w)$. Let φ_t and ψ_t the flows associated to $\dot{w} = g(w)$ and $\dot{w} = Dg(w^*)w$, respectively. Then there exists a neighbourhood, U, of w^* and an homeomorphisme, h, defined on U, with $h(w^*) = w^*$ such that it conjugates φ_t and ψ_t (that is, $h\varphi_t = \psi_t h$ for all t while the points remain in the corresponding neighbourhood).*

In other words, letting aside the transformation h (which can be regarded as a change of variables) the flows φ_t and ψ_t are the same in a neighbourhood of w^*. Notice, however, that in general h is not a differentiable map. Only the number of eigenvalues with positive real part (or with negative real part) is enough to decide if two systems of the same dimension and with hyperbolic matrices, are conjugated.

In the cases of eigenvalues with zero linear part (including, in particular, the case of zero eigenvalues) the contribution of the nonlinear part is essential to describe the behaviour close to a fixed point.

For a linear system $\dot{y} = Ay$ in $\mathbb{R}^n$, if s is the number of eigenvalues with negative real part and $u = n - s$ the number of those with positive real part, there exist two linear subspaces, E^s and E^u, having dimensions s and u , respectively (one of them can be zero) which are invariant under the linear flow. E^s and E^u are denoted as *stable subspace* and *unstable subspace*, respectively. This is the situation for the linear system $\dot{w} = Dg(w^*)w$ if w^* is an hyperbolic fixed point. For the initial equation $\dot{w} = g(w)$ there are similar objects, ${W^s}_l, {W^u}_l$, of respective dimensions s and u, called *stable and unstable local manifolds*, respectively, with the properties

$$ {W^s}_l = \{x \in U | \lim_{t\to\infty} \varphi_t(x) = w^*\}, \quad {W^u}_l = \{x \in U | \lim_{t\to-\infty} \varphi_t(x) = w^*\}, \tag{6} $$

Of course, E^s and E^u are tangent to ${W^s}_l$ and ${W^u}_l$, respectively, at w^*. By transporting the objects of of (6) under the flow we have the global manifolds:

$$ W^s = \bigcup \varphi_t({W^s}_l), \quad W^u = \cup \varphi_t({W^u}_l), $$

where the unions are done $\forall t \in \mathbb{R}$, despite for the first one is it enough to take $t < 0$ (and $t > 0$ for the second one).

In a similar way we can study the *fixed points of diffeomorphisms*: $\phi(w^*) = w^*$. If we want to study *periodic points of strict period* $k > 1$ (a point w^* is called periodic under ϕ of strict period k if $\phi^k(w^*) = w^*$, and $\phi^j(w^*) \neq w^*$ for $j = 1, ..., k-1$) it is enough to use ϕ^k instead ϕ. As the passage from differential equations to diffeomorphisms can be seen as an *exponentiation* (remember that the solution ψ_t of the linear system $\dot{x} = Ax$ is $\psi_t = e^{At}$), we shall say that a fixed point is hyperbolic if *all the eigenvalues of* $D\phi(w^*)$ *have modulus different from 1.* In this case Hartman's theorem assures the conjugacy of ϕ and $D\phi(w^*)$ on a neighbourhood of w^*. In a similar form we have stable and unstable manifolds. Note, however, that as we deal with a diffeomorphism, the points on both manifolds move on them by "jumping" and not in a continuous way.

Hartman's theorem shows the existence of W^s and W^u, but it only assures that the manifolds are continuous (remember that h is, in general, just a homeomorphism). But if $g \in \mathcal{C}^r$ or $\phi \in \mathcal{C}^r$ (in the continuous and discrete cases, respectively) the *stable/unstable invariant manifold Theorem* asserts that W^s and W^u are also of class $\mathcal{C}^r$.

It is important to remark that an hyperbolic fixed point is *persistent to perturbacions*. Indeed, reasoning for a diffeomorphism, let us consider a family of diffeomorphisms, $\phi(w, \lambda)$, depending on a parameter λ (it can be a vectorial parameter, with components $\lambda_1, ..., \lambda_p$). Assume that for $\lambda = \lambda^*$ the diffeomorphism $\phi(w, \lambda^*)$ has an hyperbolic fixed point, w^*. Hence $1 \notin SpecD_w\phi(w^*, \lambda^*)$, where $SpecA$ denotes the set of eigenvalues of the matrix A. The fixed point condition when we vary λ is

$$G(w, \lambda) = \phi(w, \lambda) - w = 0. \tag{7}$$

As $G(w^*, \lambda^*) = 0$ and $D_wG(w^*, \lambda^*) = D_w\phi(w^*, \lambda^*) - Id$ has no eigenvalue equal to zero, it is invertible. The *implicit function Theorem* tells us that, in a neighbourhood of λ^*, there exists a funtion $w = w(\lambda)$ such that $w^* = w(\lambda^*)$ and $G(w(\lambda), \lambda) = 0$. Furthermore $D_w\phi(w(\lambda), \lambda)$ is close to $D_w\phi(w(\lambda^*), \lambda^*)$ by the continuity of $D_w\phi$ and, hence, it is also hyperbolic if the neighbourhood is small enough.

Let us pass to other "simple" solutions. For an equation like (2) we can consider periodic solucions, satisfying $\varphi_t(w) = \varphi_{t+T}(w)$ for some $T > 0$. Notice that the knowledge of the solution for $t \in [0, T)$ determines it $\forall t \in \mathbb{R}$. The problem can be reduced to look for fixed points of a diffeomorphism. Indeed, let p be a point in the periodic orbit and $\mathcal{S}$ surface transversal to $g(p)$ at p. Then, the Poincaré map will be defined in a neighbourhood of p on $\mathcal{S}$ and $P(p) = p$. Note theat another possibility would be to ask for $\varphi_T(p) = p$, but it is immediate to see that the vector $\frac{d}{dt}\varphi_t(p) = g(\varphi_t(p))$ is a solution of (4). Hence $D_p\varphi_T(p)$ has 1 as eigenvalue, and we can not apply the implicit function Theorem. If the vector field is not autonomous, but T-periodic, we can look for T-periodic solutions by requiring $\varphi(T; 0, p) - p = 0$. Generically $D_p\varphi(T; 0, p)$ will not have eigenvalues equal to 1, despite other difficulties can appear (for instance, bad numerical conditioning if the field is a perturbation of an autonomous one and the part depending periodically on time is small).

Let us return to periodic orbits of an autonomous system. We say they are hyperbolic if, looking at them as fixed points of a Poincaré map, these fixed points are hyperbolic. Which Poincaré section we use is irrelevant, because using $\mathcal{S}_1$ or $\mathcal{S}_2$ one obtains conjugated differential matrices.

Consider now other *invariant objects*. A differential equation leaves invariant an object, A, if $\varphi_t(x) \in A, \forall x \in A, \forall t \in \mathbb{R}$. If $\phi^n(x) \in A, \forall x \in A, \forall n \in \mathbb{Z}$ we say that the diffeomorphism ϕ leaves A invariant. Fixed points, periodic orbits and their related stable and unstable manifolds, if they are hyperbolic, are examples of invariant objects. But other invariant objects can occur: k-dimensional tori (topologically equivalent to the product of k circles), spheres, Cantor sets, etc. An invariant object, A, es called *normally hyperbolic* under a diffeomorphism, ϕ, if for all point $x \in A$ there exists a decomposicion of $\mathbb{R}^n$ as $T_xA \oplus E_x^s \oplus E_x^u$, where T_xA is the tangent space to A at x, such that the differential of the diffeomorphism at x, $D\phi(x)$, send T_xA, E_x^s and E_x^u into $T_{\phi(x)}A$, $E_{\phi(x)}^s$ and $E_{\phi(x)}^u$, respectivelly and, furthermore, there are constants $C > 0$ and $\lambda > 0$ such that $\forall n \in \mathbb{N}$ the following holds:

a) $|D\phi^n(x)\xi| \leq Ce^{-\lambda n}|\xi|$ for all vectors $\xi \in {E_x}^s$,

b) $|D\phi^{-n}(x)\eta| \leq Ce^{-\lambda n}|\eta|$ for all vectors $\eta \in {E_x}^u$.

E^s_x gives the *stable directions* at x and E^u_x the *unstable directions* at x.

An analogous definition holds for flows of vector fields. If T_xA, E^s_x and E^u_x, which must depend on x in a continuous way, have dimensions d_A, d_s and d_u, respectivelly, then *the stable* and *unstable manifolds of* A, defined by

i) $W^s{}_A = \{x \in \mathbb{R}^n | \lim_{k\to\infty} d(\phi^k(x), A) = 0\}$,

ii) $W^u{}_A = \{x \in \mathbb{R}^n | \lim_{k\to\infty} d(\phi^{-k}(x), A) = 0\}$,

where $d(y, A)$ denotes the distance of the point y to the set A, have the following dimensions: $dim\, W^s_A = d_A + d_s$, $dim\, W^u_A = d_A + d_u$. It is clear that $d_A + d_s + d_u = n$. For instance, a normally hyperbolic periodic orbits of a flow in $\mathbb{R}^3$ has stable and unstable manifolds of dimension 2. Locally one can imagine these manifolds as cylinders.

1.4 Homoclinic phenomena. Unpredictability.

To simplify the presentation let us assume that we study planar diffeomorphisms. Consider, for instance, the so called *conservative Hénon map*, which can be written as

$$\phi_\varepsilon(x, y) = (\overline{x}, \overline{y}),$$

where

$$\begin{aligned} \overline{x} &= x + \varepsilon\overline{y}, \\ \overline{y} &= y + \varepsilon x(1 - x). \end{aligned}$$

This map depends on a parameter, ε, and it is *conservative* because the determinant *det* $D\phi_\varepsilon$ is equal to 1 for all ε, x and y, that is, it preserves the area in $\mathbb{R}^2$.

Consider now the Hamiltonian vector field $\dot{x} = y$, $\dot{y} = x(1-x)$, with energy function (or Hamiltonian) $H = y^2/2 - x^2/2 + x^3/3$. The related *phase portrait* (that is, the global picture of the orbits) is shown in figure 1.

Note that the stable and unstable manifolds of the hyperbolic fixed point ($p \equiv (0, 0)$) coincide and they are on the energy level $H = 0$. Of course, if we consider the time ε flow, φ_ε, of this vector field, the phase portrait is the same (but given a non fixed point, q, their image $\varphi_\varepsilon(q)$ is at a small distance of the point, if ε is also small). It is immediate (by using Taylor's formula) that

$$\varphi_\varepsilon(x, y) = (x + \varepsilon y + \frac{\varepsilon^2}{2}x(1 - x) + O(\varepsilon^3),\;\; y + \varepsilon x(1 - x) + \frac{\varepsilon^2}{2}(1 - 2x) + O(\varepsilon^3)).$$

In particular ϕ_ε and φ_ε coincide except by term $O(\varepsilon^3)$. Hence, we can consider the former Hamiltonian field as the *limit field* of the Hénon diffeomorphism when ε goes to zero. Then p is also a fixed point of ϕ_ε (in a general case the fixed points of ϕ_ε and φ_ε can differ in $O(\varepsilon)$). The invariant manifolds of p under ϕ_ε, $W^s{}_{p,\phi_\varepsilon}$ and $W^u{}_{p,\phi_\varepsilon}$, are

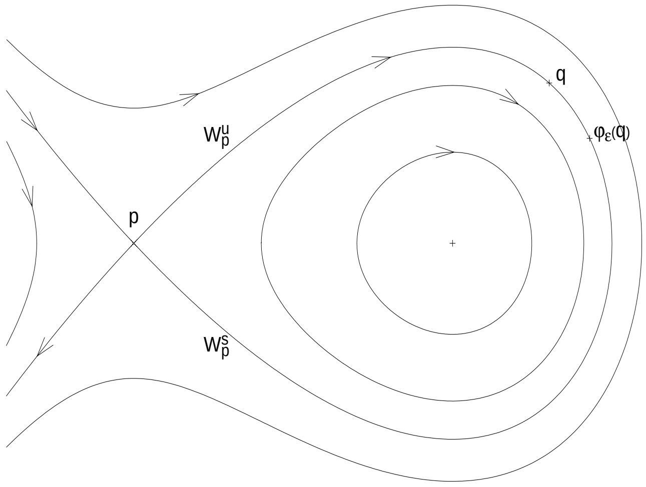

Figure 1: Phase portrait of the limit Hamiltonian flow of the Hénon map when ε goes to zero.

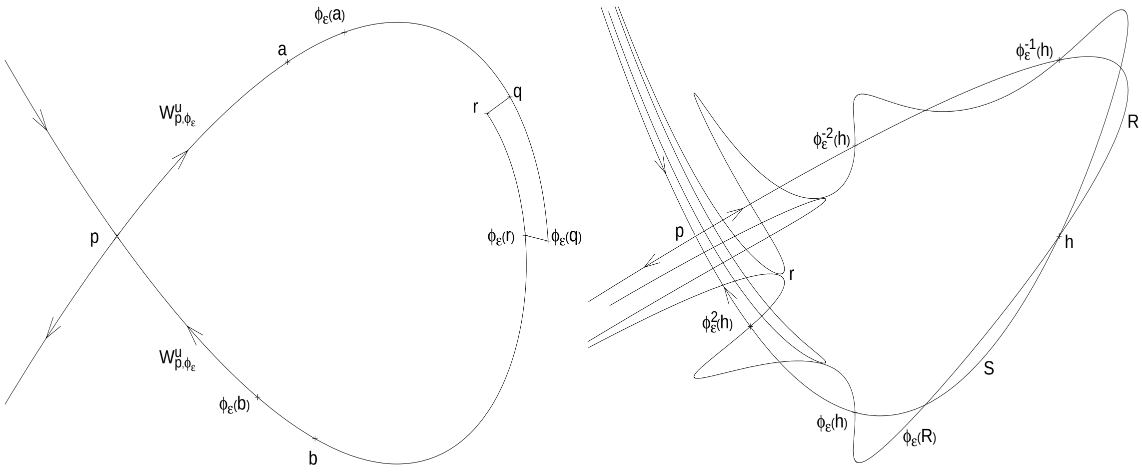

Figure 2: An illustration of the existence of homoclinic points in area preserving maps.

close to the manifolds, W^s_p and W^u_p of the flow. It is clear that W^s_{p,ϕ_ε} and W^u_{p,ϕ_ε} must intersect. Otherwise (see figure 2 left) let us consider a path as $p\,a\,q\,r\,b\,p$. The image is $p\,\phi_\varepsilon a\,\phi_\varepsilon q\;\;\phi_\varepsilon r\,\phi_\varepsilon b\,p$, where we have taken $p\,a\,q$ along W^u_{p,ϕ_ε} and $r\,b\,p$ along W^s_{p,ϕ_ε}. This leads to an increase of the area (look at the area bounded by $r\,q\,\phi_\varepsilon q\,\phi_\varepsilon r$) and, hence, to an absurdity. Therefore, either the manifolds coincide, $W^u_{p,\phi_\varepsilon} = W^s_{p,\phi_\varepsilon}$, a non generic situation (in fact, of codimension infinity, that is, one should impose infinitely many independent conditions on the diffeomorphism to be satisfied), or we are in the situation of figure 2 right. A point, h, which belongs to $W^u_{p,\phi_\varepsilon} \cap W^s_{p,\phi_\varepsilon}$ is called *homoclinic*. If, in the case of dimension 2 that we consider, W^u_{p,ϕ_ε} and W^s_{p,ϕ_ε} intersect in h making a non zero angle (as it is seen in figure 2 right), we say that h is a *transversal homoclinic point*. In some cases it can happen that, on a neighbourhood of h, one invariant manifold "crosses to the other side" of the other manifold, but the angle is zero. (Think about the manifolds having an odd order tangency at h). Then it is said that h is *topologically transversal*, and, from a qualitative point of view, this case is similar to the transversal one. Of course, it is also possible that there exist points, as the point r shown in figure 2 right, where a *homoclinic tangency* occurs. By changing ε the point r can *disappear or bifurcate* in several points, while *h persists* (and it is locally unique) under perturbation. Furthermore, the different "lobes", as $\mathcal{R}$, $\mathcal{S}$, $\phi_\varepsilon\mathcal{R}$, etc, have all the same area.

To see the dynamical consequences of this behaviour, we shall use a family of maps different from the Hénon one, the family being cubic instead of quadratic:

$$\phi_\varepsilon(x,y) = (\overline{x},\overline{y}),$$

where

$$\begin{aligned}\overline{x} &= x + \varepsilon\overline{y},\\ \overline{y} &= y + \varepsilon(x - x^3).\end{aligned}$$

The situation is similar to the one of figure 2, but now we have two symmetric "loops" instead of one (see figure 3, which shows to the right a magnification of the central part by a factor of 10).

Let us study the behaviour of nearby points, as A and B, close also to a branch, $W^{u,+}_p$, of the unstable manifold of the fixed point $p \equiv (0,0)$. Let us simply denote by ϕ the map ϕ_ε for a fixed value of ε (for instance, $\varepsilon = 0.55$, the actual value used for the illustration). Using a convenient iterate, ϕ^k, both points $\phi^k(A)$ and $\phi^k(B)$ can be seen on different sides of $W^{s,+}_p$ (see figure 3). Of course, this means that they already were on different sides since the very beginning, but this is much hard to see. The successives iterates will take them again close to p, being close to $W^{s,+}_p$ but on *different sides*. Hence, after a passage close to p one of them will follow closely the branch $W^{u,+}_p$, while the other will follow close to $W^{u,-}_p$. It is clear that both the images of A and B, will return once and again close to p. At every passage small deviations will allow to

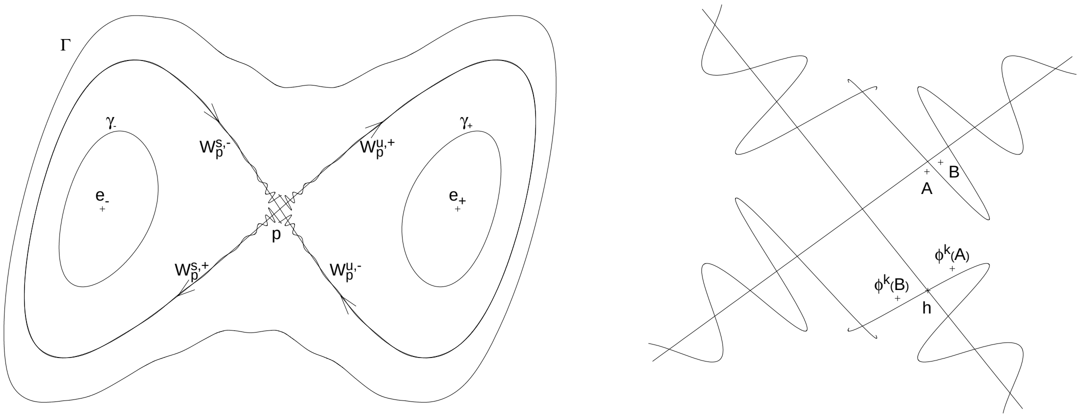

Figure 3: Phase portrait of the cubic area preserving diffeomorphism showing invariant curves and transversal homoclinic points. The right part is a magnification of a neighbourhood of the hyperbolic fixed point by a factor of 10.

"choose" which side of W_p^s they follow. We remark that, in the situation of figure 3 the following result can be proved:

Theorem. *For a given value of $\varepsilon > 0$, $\varepsilon < \varepsilon_0$, fixed, and using the map ϕ_ε, there exists $N \in \mathbb{N}$ such that, per every couple of sequences (doubly infinite, that is, infinite at both sides) $(..., n_{-3}, n_{-2}, n_{-1}, n_0, n_1, ...)$, $(..., s_{-3}, s_{-2}, s_{-1}, s_0, s_1, ...)$ where $n_j \geq N \, \forall j \in \mathbb{Z}$ and s_j is a sign (+ or -), there is at least an initial point, A, close to p, such that under iteration it goes to the right if s_0 is $+$ and to the left if s_0 is $-$, and which returns at a local minimum distance to p after n_0 iterates; then it goes to the right or to the left according to the sign of s_1, returning again to a local minimum distance to p after n_1 additional iterates, etc. The same happens by using the inverse iteration with signs and numbers of iterates given by s_{-1} and n_{-1}; s_{-2} and n_{-2}, etc.*

Of course, this tells us that we can have all kinds of possible behaviour. This type of dynamics is called *quasirandom*, because any sequence $(..., n_{-3}, n_{-2}, n_{-1}, n_0, n_1, , , ,)$, $(..., s_{-3}, s_{-2}, s_{-1}, s_0, s_1, ...)$ *can be realized*, as if we could choose at random to go right or left every time the iterates pass close to p. In other words: the behaviour is *unpredictible* despite the system is deterministic. This can be a little bit surprising and requires a more detailed explanation.

First, and due to the determinism, it is clear that, given any initial point, A, the orbit is uniquely determined, and the related sequences of n and s is unique. Let us examine more closely the passages near p. There we can approximate ϕ by the linear part, the matrix $\begin{pmatrix} 1+\varepsilon^2 & \varepsilon \\ \varepsilon & 1 \end{pmatrix}$, which has an eigenvalue $\lambda_u = 1 + \varepsilon\sqrt{1+\varepsilon^2/4} + \varepsilon^2/2$ such that the corresponding eigenvector is tangent to W_p^u. In a similar way, there is an eigenvalue $\lambda_s = 1/\lambda_u$ with eigenvector tangent to W_p^s. Close to p, every time we iterate under ϕ the distance between points is increased by a factor λ_u in the direction of the unstable manifold and it is compressed by λ_s in the direction of the stable one. Doing k iterates on a prefixed neighbourhood of p (for instance, the disk of radius 0.1) the unstable component will be multiplied, approximately, by ${\lambda_u}^k$. Using, for instance, the mentioned value $\varepsilon = 0.55$ and $k = 100$ the amplification factor is close to $1.2 \cdot 10^{24}$. But the position of A is not exactly known. Making simulations in a computer one has a rounding error (with a relative value of 10^{-16}, say). Studying a natural phenomenon, if A denotes the state at a given time, we shall have error in the measurements of the position, say. Hence, the amplification of the distances along the unstable direction produces that we loose the control on the behaviour. It could happen that the iterates far away from p (that is, outside the selected neighbourhood) cancel exactly the expansions and compressions produced inside it. This really happens if instead of ϕ_ε we use φ_ε, the time ε flow described above. But this *integrability* is against the existence of a transversal homoclinic point as h in figure 3.

The studied map is area preserving and close, if ε is small, to the flow φ_ε. The fixed points e_+ and e_-, having coordinates $(1, 0)$ and $(-1, 0)$, respectively are of elliptic type. Locally φ_ε (and ϕ_ε)) behaves like a rotation. We recall that, for an area preserving map, ϕ, if an eigenvalue of $D\phi(x^*)$ is λ, where we assume x^* to be a fixed point of ϕ, then the other eigenvalue is $1/\lambda$. The fixed point is called *elliptic* if λ is a complex number of modulus 1, different from ± 1.

Under the flow φ_ε the limit period when going to $e_{\pm e}$, is $\pi\sqrt{2}$. When approaching the separatrix (the level $H = 0$) the period increases monotonically to infinity. The *rotation number* is defined, on these orbits, as the frequency: $2\pi/period$. As ϕ_ε is a relatively small perturbation ($O(\varepsilon^2)$) of φ_ε and φ_ε is a *twist map* (the derivative of the rotation number with respect to the energy is different from zero), Moser's twist Theorem (or, as more general, the Kolmogorov-Arnol'd-Moser, *KAM* Theorem) asserts that there exist closed curves invariant under ϕ_ε. Figure 3 shows three of these curves, as $\gamma_\pm$ around $e_\pm$, respectively, and Γ in the outer part. This implies that the successive iterates of points like A are confined. This is, of course, true if ε is small enough. think that in the present example it is required to have $|\varepsilon| < 0.56...$. By plotting iterates of a suitable point close to p, it seems that the iterates cover a domain of positive measure. The natural question is whether *the closure of the orbit has positive measure or not.* This is an open question up to now.

Some suitable references for this introduction are listed below.

- Arnold, V.I.:*Geometric theory of ordinary differential equations*, Springer 1982.

- Guckenheimer, J., Holmes, P.:*Nonlinear oscillations, dynamical systems and bifurcations of vector fields*, Applied Math. Sci. 42, Springer 1983.

- Palis, J., de Melo, W.:*Geometric theory of dynamical systems: an introduction*, Springer 1982.

- Palis, J., Takens, F.:*Hiperbolicity and sensitive chaotic dynamics at homoclinic bifurcations*, Cambridge Univ. Press 1993.
- Wiggins, S.:*Global bifurcations and chaos*, Applied Math. Sci. 73 Springer 1988.

In what follows we shall pass to higher dimension to be able to deal with models closer to the requirements of the applications. The situation, from a theoretical point of view, is much less clear than in the case of area preserving maps (or, equivalently, Hamiltonians with two degrees of freedom). From a practical point of view, the difficulties of the numerical computations and of the representations are increased, but still there are lots of methods which can be used. Some of them will be presented in the forthcoming sections.

2 The invariant objects: periodic orbits, invariant tori, invariant stable and unstable manifolds, centre manifolds

Celestial Mechanics is a part of Analytical Mechanics dealing with the study of the motion of natural and artificial bodies under the action of gravitation. Usually the number of interacting bodies is considered not too large. Other problems, like the dynamics of galaxies and clusters, have usually a different approach. The study of the motion of artificial bodies has some peculiarities (design of missions by means of suitable manoeuvres) and is sometimes denoted as Astrodynamics.

Celestial Mechanics and Astrodynamics can be seen under the point of view of Dynamical Systems. We shall consider, in what follows, that they admit a Hamiltonian formulation, despite in many cases one should allow for non conservative forces. Hence we shall assume that we are given a Hamiltonian $H = H(q, p)$, where q and p denote positions and momenta. In principle and for simplicity, we shall assume that H is an analytical function defined on a $2n$–dimensional space, that typically will be $\mathbb{R}^n$ with the obvious exceptions to avoid collisions of bodies. To present the main ideas we shall restrict our exposition to the interaction of punctual masses under gravitational attraction. The reader can easily adapt most of the methods to other classes of problems (including non conservative ones for several topics). Hence the equations of motion are

$$\frac{d}{dt}q_i = \dot{q}_i = \frac{\partial H}{\partial p_i}, \;\; \frac{d}{dt}p_i = \dot{p}_i = -\frac{\partial H}{\partial q_i}, \;\; i = 1, \ldots, n.$$

The number n is known as the *number of degrees of freedom*. The $2n$–dimensional space, $\mathcal{E}$, where the variables q and p live is known as the *phase space*. A key idea of the theory of Dynamical Systems is to look at any problem concerning the dynamics as a *geometrical problem* in the phase space. Hence, we look always for objects which have an intrinsic geometrical meaning, independent of the actual parametrizations used to describe them.

From a geometrical point of view a key point consists in trying to understand the structure of the phase space. We can find in it fixed points, periodic orbits and invariant tori. Some of these objects can be (at least partially) of *hyperbolic* character. The related stable and unstable manifolds form a kind of network of connections (the so called *homoclinic tangle*), which together with the previous invariant objects constitute the *skeleton* of the system. The dynamics follows closely the "paths" marked by these connections. The objects which are of purely *elliptic* type, under suitable conditions on the related frequencies, have the nearby phase space "almost foliated" by invariant tori. Here "almost" means that the we have a set of tori depending on parameters (say, amplitudes or "actions") and the parameters range on a Cantorian set of big relative measure. The "gaps" in the Cantor set are due to resonances, but if these resonances are of high order the gaps can be very small. In them there is a mild lack of predictability (or, if you want, some "chaos"). Notice that these gaps can be connected, leading to a *diffusion* of the action variables, but at a very slow rate. Other objects are partially elliptic and partially hyperbolic. Among them we can consider the ones having just an hyperbolic plane, that is, only one positive and one negative eigenvalues (with the same modulus because of the Hamiltonian character). These objects have *centre manifolds* of codimension 2, and the stable and unstable manifolds of these centre manifolds have codimension 1. They are ideal candidates to act as "barriers" or *confiners* to make difficult the escape of points. In fact these manifolds do not prevent escape: stable and unstable manifolds do not match but, generically, they intersect transversally. The manifolds are highly folded and this, together with small angles at the intersections, produces that the particles of the phase space have to wander for a long time before finding the way out.

The simplest solutions of the equations are the *fixed points*, where the vector field given by the right hand side of the equations above becomes zero. The next simpler solutions are the *periodic solutions*. If, for shortness, we denote by z the vector containing all the components of q and p, then a periodic solution if of the form $z(t) = w(\theta)$, where θ is an angle in $\mathbb{S}^1$ moving with time as $\theta = \omega t + \theta_0$, $\omega \in \mathbb{R}$ being the *frequency* and $\theta_0 \in \mathbb{S}^1$ being an *initial phase*. In a similar way we can consider *quasiperiodic solutions*, formally expressed as the periodic ones, but θ belonging now to $\mathbb{T}^k$, a k–dimensional torus, $\omega \in \mathbb{R}^k$ being a vector of frequencies and the initial phase, θ_0, being also in $\mathbb{T}^k$. The frequencies in the vector ω are denoted as the *basic frequencies* of the quasiperiodic solution. We shall assume that they satisfy a *non resonance condition*, that is,

$$(m, \omega) \neq 0, \ \ \forall m \in \mathbb{Z}^k \setminus \{0\}.$$

Here (,) denotes the scalar product. Otherwise the frequencies are resonant and some of them can be put as a function of the remaining ones, i.e., the number of basic frequencies is less than k.

Now consider Hamiltonians *close to integrable.* An n–degrees of freedom Hamiltonian is called *integrable* in the Liouville sense if it has n *functionally independent*, except, perhaps, on some manifolds of lower dimension, *first integrals*, $F_i,\ i = 1, \ldots, n$, one of them being, eventually, the Hamiltonian H itself, and such that they are *in involution.* That is, the *Poisson bracket*

$$[F_i, F_j] = \sum_{k=1}^{n} \frac{\partial F_i}{\partial q_k}\frac{\partial F_j}{\partial p_k} - \frac{\partial F_i}{\partial p_k}\frac{\partial F_j}{\partial q_k}$$

is zero for all choices of i and j. Then, consider the level surfaces of the first integrals,

$$\mathcal{M}_c = \{(q,p) \in \mathcal{E} | F_i(q,p) = c_i,\ i = 1, \ldots, n\},$$

the vector c having components c_i. If $\mathcal{M}_c$ is compact, then the celebrated Liouville–Arnol'd theorem assures that it is diffeomorphic to an n–dimensional torus. There are many other definitions of integrability (in Hamiltonian and other problems). The ones having an interesting dynamical meaning are those who guarantee a nice behavior of the solutions, so that there is no impredictability of the motion.

An integrable Hamiltonian, H_0, in the sense above can be put in *action–angle* variables (I, φ), and then $H_0 = H_0(I)$. Hence, the solutions are trivial:

$$I(t) = I(0),\ \ \varphi(t) = \varphi(0) + \omega t,$$

where the frequencies ω are given by $\omega_j = \omega_j(I) = \frac{\partial H_0}{\partial I_j}$. There are open sets of the phase space foliated by invariant tori. If we look now at a perturbation

$$H(I, \varphi) = H_0(I) + \varepsilon H_1(I, \varphi, \varepsilon)$$

of the Hamiltonian H_0, under some generic conditions, ensuring that the frequencies of the "unperturbed" Hamiltonian *do* change with the actions (i.e., $\frac{\partial \omega(I)}{\partial I}$ is an invertible matrix), the well known KAM theorem (Kolmogorov–Arnol'd–Moser) assures that the perturbed Hamiltonian still has most of the invariant tori, but they do not constitute a foliation at all. Between any two tori there are extremely narrow domains where some amount of "chaoticity" takes place. It can be completely irrelevant from the physical point of view and numerically undetectable, unless one is willing to do computations with a large number of digits. The "robust" tori, surviving the perturbation for ε small enough, are those whose frequencies satisfy some *Diophantine condition.* More concretely, positive values of c and τ must exist so that

$$|(m, \omega)| > c||m||^{-\tau},\ \forall m \in \mathbb{Z}^n \setminus \{0\}.$$

If $\tau > n - 1$ the set of frequencies ω for which this condition is not satisfied for any $c > 0$, has zero Lebesgue measure.

In some problems we have to allow for time depending Hamiltonians, $H = H(q, p, t)$. The cases that we shall consider here come from the consideration of the motion of a small particle under the action of more massive bodies. The particle does not affect the motion of the other bodies. So their motion can be taken as a known function of time. Usually the periodic or quasiperiodic approximation are good enough models. Then the dependence of H on t is periodic or quasiperiodic. The autonomous character of the Hamiltonian can be recovered by the usual method to add new variables. For instance, if $H(q,p,t)$ is a Hamiltonian depending on t in a quasiperiodic way through $\theta \in \mathbb{T}^d$ with vector of basic frequencies $\omega \in \mathbb{R}^d$, we can introduce a new Hamiltonian

$$\mathcal{H}(q,p,\theta,J) = H(q,p,\theta) + (\omega, J),$$

where $J \in \mathbb{R}^d$ are additional momenta, canonically conjugated to the angles θ. Of course, then we have $\dot{\theta}_j = \frac{\partial \mathcal{H}}{\partial J_j} = \omega_j,\ \ j = 1,\ldots,d$. The formulation is reduced to the autonomous case, but there is a main difference. The derivatives of the new Hamiltonian with respect to these new momenta are constant (i.e., they are the "external frequencies") and this gives some kind of degeneracy, or *lack of free actions*, when trying to apply some KAM–like results.

When a fixed point has been computed, the first thing to study is the linear stability. This is easily obtained from the eigenvalues of the linear part of the vector field at the point. If, as before, z denotes all the variables of the Hamiltonian, we have $\dot{z} = X_H(z)$, where X_H denotes the Hamiltonian vector field. Then, if the fixed point is denoted as z^*, one has to look for the eigenvalues of $DX_H(z^*)$. It is well known that, due to the infinitesimal symplectic character of $DX_H(z^*)$, if λ is an eigenvalue, so is $-\lambda$, and if λ is not real, also $\pm\bar{\lambda}$ are eigenvalues, where $\bar{\lambda}$ denotes the complex conjugate of λ. Furthermore, if 0 is an eigenvalue, then is has even multiplicity.

If we only look at the linear behavior around z^*, we can classify it according to the number of eigenvalues of $DX_H(z^*)$ having positive, negative or zero real part. The related eigenvectors generate linear subspaces known as the unstable, stable and central subspaces, respectively. Of course, the dimension of the stable and unstable subspaces are the same. The stable and unstable eigenvalues give the *hyperbolic part* of the local behavior, that is, the part associated to eigenvalues with non zero real part. The non zero eigenvalues with zero real part give rise to the *elliptic part* of the local behavior. The name *parabolic part* can be reserved for the subspaces associated to zero eigenvalues.

When we turn to the nonlinear behavior, a generalization of the previous description holds. There are *local stable and unstable manifolds*, which are invariant manifolds, under the flow of X_H, tangent to the corresponding linear subspaces. They are also characterized as the set of points, in a given neighborhood of z^*, such that under the flow for positive (respectively, negative) time go to z^* without leaving the neighborhood. By globalization under the flow (that is, by looking for the images of the local stable/unstable manifolds for all time for which the flow is defined) they give the *global stable and unstable manifolds*. Concerning regularity, as we assumed the Hamiltonian to be analytic, the stable and unstable manifolds are also analytic.

The central part is more subtle. As it follows from the formal computation, there is always a *formal* centre manifold. That is, a manifold which can be expressed, for instance, as the graph of a function given by a formal power series. But, despite the Hamiltonian is assumed to be analytic, the centre manifold is, in general, not more than $\mathcal{C}^\infty$. Hence, we can add to this function any $\mathcal{C}^\infty$–flat function and it will be also a formal centre manifold. We are faced here to a problem of *non uniqueness*. Of course, if the formal power series turns out to be convergent, then there exists an *unique* $\mathcal{C}^\omega$ *centre manifold*. Another problem is the *globalization of the centre manifold*. In general it is a hard problem to decide about "the boundary of the centre manifold". I know examples where the centre manifold "ends" or "losses dimension" at some separatrix, but also of examples where it "increases dimension" by "absorbing" some directions that were previously hyperbolic. These changes are related to *global bifurcations* and they are typically hard to study but very relevant for the geometry of the phase space (and, hence, for the dynamics).

Taking now a centre manifold as an invariant object, we can look, if it do not has full dimension, to the stable/unstable manifolds of this centre manifold. If we are dealing with the centre manifold of a fixed point, for instance, the stable manifold of the centre manifold is usually denoted as *centre–stable manifold* of the point. In a similar way we can talk about *centre–unstable manifolds*. Assume that the hyperbolic part has dimension 2, that is, only one stable and one unstable directions exist. Then the centre–stable and centre–unstable manifolds play a key role in some aspects of the dynamics, as sketched above, because they have *codimension 1*.

Now we turn to the behaviour around periodic and quasiperiodic solutions. As usual, we can turn the orbit into a fixed point by introducing new (time dependent) variables. Let $z(t)$ a solution. Then the change $w = z - z(t)$ converts the solution into a fixed point. Of course, this change is symplectic and, hence, it keeps the Hamiltonian character of the equations of motion. The problem is, however, that even if the the initial Hamiltonian is autonomous, the new one, $H(w + z(t)$, will have the same dependence with respect to time as $z(t)$ has. Consider, first the case of a periodic solution.

The point $w = 0$ being fixed, the Hamiltonian around it has a power series expansion starting with terms of degree 2. We shall denote as H_2 these terms. Of course, all the homogeneous parts of the series have coefficients which depend periodically on t, with the same period of $z(t)$. If we linearize around $w = 0$, only the terms coming from H_2 are relevant. We know that this linear equation has periodic coefficients in t. It is well known that, by the Floquet theory, there is a linear change of variables, with the same period as the coefficients, which turns the linear equation to constant coefficients. This change can be taken to be symplectic, to preserve the Hamiltonian. Hence, the local character of the fixed point $w = 0$ is well defined. We can compute the eigenvalues and decide about the dimensions of the invariant manifolds. One should take into account that, if the initial Hamiltonian is autonomous, there will be always a zero eigenvalue associated to the phase shift along the periodic orbit. It must appear with even multiplicity. Another eigenvalue zero is associated to the conservation of the energy. Generically, only these two zeros appear as eigenvalues, but there are cases where the multiplicity can be higher (e.g., in the case of existence of additional first integrals). Also generically, when there are only two zero eigenvalues, the matrix of constant coefficients of the linearized equations reduced to the related subspace do not diagonalizes. It has a Jordan block structure. It follows easily from the implicit function theorem that there exist a *natural continuation* of the periodic orbit to a *family of periodic orbits*. Typically this family is, locally, parametrized by the energy. The out of diagonal element in the Jordan block is associated to the *variation of the period with respect to the energy level*.

It is useful to consider, to study the vicinity of a periodic orbit, a *transversal section* or *Poincaré section*. Then the dimension of the phase space is reduced by 1, and the flow is replaced by a *discrete dynamical system*, the so called *Poincaré map*, to be denoted as P. We can look at it when all the (local) levels of energy are considered, or when we fix the level at the one of the periodic orbit. In this last case the dimension will be reduced by 2, and the Poincaré map turns out to be a symplectic map on the corresponding space. Let us denote as $\bar{w}$ a local variable around 0 for this $2n-2$ Poincaré map and keep the same name, P, for the map. We can look at the local properties of P around $\bar{w}=0$. Now the stable, unstable and central directions are associated, respectively, to the eigenvalues of $DP(0)$ with modulus less than 1, greater than 1 or equal to 1. If the full $(2n-1)$ Poincaré map is considered, one should increase the central dimension by 1. To return to the behavior around the full periodic orbit, one should transport what happens in the Poincaré section under the flow. Again the central dimension is increased in an additional unit.

The quasiperiodic case is not so easy. The linearized equations can not be reduced to constant coefficients, in general. Anyway, if the linear system can be put as a constant coefficients one plus a small perturbation, depending on time quasiperiodically, then the reduction is possible for a relativelly large (but Cantorian) set of values of the perturbation parameter, provided some non degeneracy conditions hold. To avoid to require it at each sentence, we shall assume, from now on, that a *reducibility condition holds*. That is, that there exist a linear change of variables, depending quasiperiodically on time with the same set of basic frequencies as the $z(t)$ solution considered, such that it reduces the linearized equations to constant coefficients. There are ways to check, approximately, if this condition is satisfied. Under this assumption, the local character of the quasiperiodic solution is well defined. We can talk about hyperbolicity, stable, unstable and central directions, etc. Just note that if the dimension of the torus is k, then there are $2k$ eigenvalues equal to zero. One half of them is associated to the *phases shift* along the angles of the torus. The other half is associated to the formal preservation of the associated momenta. As mentioned before, if we are given an $n-1$–dimensional torus and it turns out to have just one stable and one unstable directions, the centre manifold associated to it has dimension $2n-2$ (think about a family of $n-1$–dimensional tori, depending on $n-1$ parameters on a Cantor set with very narrow gaps) and, hence, the centre–stable and centre–unstable manifolds have codimension 1.

Some of the usual techniques to deal with problems in Celestial Mechanics can be considered inside perturbation theory. An approximation to the problem can be integrable, so that the solutions can be expressed in a simple form (at least in theory!) and we can look for solutions of the full problem expanded in power series of some parameter. This requires analyticity of the equations of motion and their dependence with respect to parameters. The solution of the integrable system gives the zero order terms of this expansion. It can be also possible to try to perform changes of variable such that the problem is reduced to a simpler form, hopefully integrable. This is not possible, of course, but averaging methods or normal form techniques often succeed in carrying out such a reduction at the formal level. The lack of convergence of these procedures, due to small divisors problems in most of the cases, asks for bounds on the errors done when several steps of an iterative process are performed. Furthermore, one can derive information about the number of steps to be used so that the estimate of the total error is minimized. Typically this minimum is exponentially small in the small parameter involved in the expansions. More refined information requires the use of time as a complex variable.

However, most of the problems in Celestial Mechanics are beyond the present limits of the purely analytical techniques. Quite often one has to rely on numerical simulations to understand the behaviour of the solutions of some problem. On the other hand, purely numerical methods are not enough when the number of variables and parameters involved is too large. I consider that, at the present epoch, reliable and complete enough simulations can be done with at most 6 or 7 total variables. A combination of symbolic and numerical techniques, guided by careful analytical studies and the suitable geometric insight, can give very good and useful results.

In what follows we shall present some methods for the computation of different objects. They are presented through concrete examples, worked out in detail. The examples concern the motion in the 3D Restricted Three Body Problem (RTBP) around relative equilibrium (or *libration points*) of collinear and triangular type. The study is useful both to understand the dynamics of small bodies in the solar system and for the design of spacecraft missions.

At the end of these notes there appear some references for most of the topics presented and some other useful complements.

3 Global Description of the Orbits Near the L_2 Point of the Earth–Moon System in the RTBP model

Part of the methodology presented in this section and in the next one was done in collaboration with G. Gómez, À. Jorba and J. Masdemont under contracts of the European Space Agency. It was applied to the L_1 point of the RTBP in the Sun–Earth case and to the L_4 and L_5 points of the Earth–Moon system. Later on has been applied to some improved models of the physical problem and, finally, to the best available model: the one derived from the numerical JPL ephemeris.

Let us consider the spatial Restricted Three Body Problem, that is the motion of a massless particle (to be taken as a spacecraft) under the gravitational influence of two massive bodies (to be taken as Earth and Moon). Last two are assumed to be on circular orbits around their center of masses. Using a rotating system of reference with period the one of the circular orbits and center at the mass center, the two massive bodies can be considered at rest. Scaling masses and distances, the sum of the masses of Earth and Moon and their distance can be taken equal to one. Then mass of the Moon is then the parameter $\mu_{\rm Moon}$ equal to 0.012150582.

In the rotating system of reference (synodical system) there are 5 equilibrium points (or libration points in the classical terminology). We are interested in the translunar libration point, which is aligned with Earth and Moon and placed beyond the Moon. L_1 denotes the equilibrium point between the Earth and the Moon, which is close to the Moon.

In the spatial RTBP the L_1 and L_2 libration points have a linear behaviour of the type saddle $\times$ center $\times$ center. Under generic assumptions there exist nearby two dimensional tori. Because of the small divisors problems, on those tori the frequencies should satisfy a diophantine condition. If we are interested in short time intervals the small divisors problems do not show up unless they are associated to low order resonances. Let ω_1, ω_2 be the two basic frequencies at L_i, i=1, 2. For small values of μ the frequencies are rather close. For the limit Hill's problem, $\mu = 0$, they are $(28^{1/2}-1)^{1/2}$ and 2, respectively. For $\mu = \mu_{\rm Moon}$, at the L_2 point, the values of ω_1 and ω_2 are near 1.86264588 and 1.78617616, and beyond the 1 to 1 resonance the next strong resonance appears only at order 47. A very strong resonance appears at order 8398.

When the amplitudes of the orbit change, so do the frequencies. It is known that the planar periodic Lyapunov orbits emanating from the libration point, bifurcate to halo orbits when the x amplitude (x is in the direction of the primaries) is roughly 0.1845, when the unit of distance is the one from the small mass to the libration point. In the present case this equals the value $\gamma = 0.16783273$ times the Earth–Moon distance. Hence this is the distance corresponding to the 1 to 1 resonance giving rise to the halo orbits. If we keep our study in a smaller region the effect of this resonance is not too strong. Higher order resonances have small amplitude and long period. Therefore we can try to obtain the invariant tori formally to some order (less than 47). In the concrete example presented below the order has been kept to 45.

Cutting the expressions to this order we obtain a foliation by invariant tori in some region around the equilibrium point. For some of the amplitudes the tori certainly do not exist, but the small size of the related stochastic zone makes this fact irrelevant for short time interval applications, in our case a spacecraft running during a few years.

3.1 The Equations of Motion

We consider the following system of reference: The origin is located at L_2. The positive x axis is directed from L_2 away from the smaller primary. The y axis is in the plane of sidereal motion of the primaries at $\pi/2$ from the x axis in counterclockwise sense. The z axis completes a positively oriented coordinate system. Let γ be the distance between m_2 and L_2, where m_2 denotes the position of the smaller primary. γ will be the new unit of distance. The unit of time is such that the period of revolution of the primaries in the sidereal system is 2π. Let μ be the mass of the smaller primary (in our case the Moon as given above) and $1-\mu$ the one of the bigger primary (the Earth in our application). The value of γ is obtained from the Euler quintic equation:

$$\gamma^5 + (3-\mu)\gamma^4 + (3-2\mu)\gamma^3 - \mu\gamma^2 - 2\mu\gamma - \mu = 0.$$

It can be solved by Newton method starting at $(\mu/3)^{1/3}$.

Let X, Y, Z be the usual synodical coordinates centered at the center of mass. The equations of motion are

$$\ddot{X} - 2\dot{Y} = \Omega_X, \qquad \ddot{Y} + 2\dot{X} = \Omega_Y, \qquad \ddot{Z} = \Omega_Z, \tag{8}$$

where

$$\Omega = \frac{1}{2}(X^2+Y^2) + (1-\mu)r_1^{-1} + \mu r_2^{-1},$$

being

$$r_1^2 = (X-\mu)^2 + Y^2 + Z^2, \qquad r_2^2 = (X-\mu+1)^2 + Y^2 + Z^2.$$

As we have described we make the change of coordinates

$$x = -\frac{1}{\gamma}(X-\mu+1+\gamma), \qquad y = -\frac{1}{\gamma}Y, \qquad z = \frac{1}{\gamma}Z. \tag{9}$$

Then

$$r_1^{-1} = (1-\gamma)^{-1}(1 + \frac{2\gamma}{1-\gamma}x + \frac{\gamma^2}{(1-\gamma)^2}\rho^2)^{-1/2} = (1-\gamma)^{-1}\sum_{n\geq 0}(\frac{\gamma\rho}{1-\gamma})^n(-1)^n P_n(\frac{x}{\rho}),$$

$$r_2^{-1} = \gamma^{-1}(1-2x+\rho^2)^{-1/2} = \gamma^{-1}\sum_{n\geq 0}\rho^n P_n(\frac{x}{\rho}),$$

where $\rho^2 = x^2+y^2+z^2$ and P_n denote the Legendre polynomials.

Let

$$c_n = \gamma^{-3}\left(\mu + (-1)^n(1-\mu)(\frac{\gamma}{1-\gamma})^{n+1}\right). \tag{10}$$

auxiliary coefficients depending only on μ. From (10) it is easily seen that, in the Hill's case, when μ tends to zero, $c_2 = 4$ and $c_k = 3\times(-1)^k$ if $k>2$. Then the equations of motion are written as

$$\begin{aligned}
\ddot{x} - 2\dot{y} - (1+2c_2)x &= \frac{\partial}{\partial x}\sum_{n\geq 3} c_n\rho^n P_n(\frac{x}{\rho}),\\
\ddot{y} + 2\dot{x} + (c_2-1)y &= \frac{\partial}{\partial y}\sum_{n\geq 3} c_n\rho^n P_n(\frac{x}{\rho}),\\
\ddot{z} + c_2 z &= \frac{\partial}{\partial z}\sum_{n\geq 3} c_n\rho^n P_n(\frac{x}{\rho}).
\end{aligned} \tag{11}$$

We recall also, for further use, that (8) can be written in Hamiltonian form. Introducing

$$P_X = \dot{X} - Y, \qquad P_Y = \dot{Y} + X, \qquad P_Z = \dot{Z},$$

the Hamiltonian is

$$\begin{aligned}
H &= \frac{1}{2}(P_X^2+P_Y^2+P_Z^2) + YP_X - XP_Y - (1-\mu)r_1^{-1} - \mu r_2^{-1} =\\
&= \frac{1}{2}(\dot{X}^2+\dot{Y}^2+\dot{Z}^2) - \Omega.
\end{aligned} \tag{12}$$

The equations (11) can also be put in Hamiltonian form. Let $p_x = \dot{x}-y$, $p_y = \dot{y}+x$, $p_z = \dot{z}$. Then the corresponding Hamiltonian is

$$K = \frac{1}{2}\left(p_x^2+p_y^2+p_z^2\right) + yp_x - xp_y - \sum_{n\geq 2} c_n\rho^n P_n\left(\frac{x}{\rho}\right). \tag{13}$$

The Hamiltonians H and K are related by

$$H = K\gamma^2 - \frac{1}{2}(1-\gamma-\mu)^2 - \frac{\mu}{\gamma} - \frac{1-\mu}{1-\gamma}.$$

3.2 Formal Series Solutions

If we skip in (11) the right hand terms (of degree ≥ 2) the solution of the system, restricted to the center manifold (i.e., deleting the exponentially increasing or decreasing terms) is

$$x = \alpha\cos(\omega_1\tau+\phi_1), \qquad y = -\overline{k}\alpha\sin(\omega_1\tau+\phi_1), \qquad z = \beta\cos(\omega_2\tau+\phi_2), \tag{14}$$

with

$$\overline{k} = (\omega_1^2 + 2c_2 + 1)/(2\omega_1).$$

The parameters α, β are arbitrary amplitudes and ϕ_1, ϕ_2 are arbitrary phases. The frequencies in (14) are $\omega_1 = ((2 - c_2 + (9c_2^2 - 8c_2)^{1/2})/2)^{1/2}$, $\omega_2 = c_2^{1/2}$. As (11) is an autonomous system we can take the origin of time such that $\phi_1 = 0$. Furthermore we denote $\omega_1 \tau$ and $\omega_2 \tau + \phi_2$ as θ_1 and θ_2, respectively. When the right hand terms of (11) are considered we should allow for varying frequencies depending on the amplitudes, according to the Lindstedt–Poincaré method to avoid secular terms. Then θ_1 will be now $d\tau$ and θ_2 will be $f\tau + \phi_2$, where $d = \omega_1 + O(\alpha, \beta)$, $f = \omega_2 + O(\alpha, \beta)$.

We look for solutions in complex exponential form of the type

$$x = \sum x_{i,j,k,m} \alpha^i \beta^j \gamma_1^k \gamma_2^m, \quad y = \sqrt{-1} \sum y_{i,j,k,m} \alpha^i \beta^j \gamma_1^k \gamma_2^m, \quad z = \sum z_{i,j,k,m} \alpha^i \beta^j \gamma_1^k \gamma_2^m, \tag{15}$$

where $\gamma_s = \exp(\sqrt{-1}\theta_s)$, $s = 1, 2$.

Due to the symmetries of the problem the coefficients in (15) satisfy the relations:

1. $i, j \geq 0, i + j \geq 1, \mid k \mid \leq i, \mid m \mid \leq j, i - k \equiv 0 \pmod 2$, $j - m \equiv 0 \pmod 2$ in all the cases,

2. $x_{i,j,-k,-m} = x_{i,j,k,m}$, $y_{i,j,-k,-m} = -y_{i,j,k,m}$, $z_{i,j,-k,-m} = z_{i,j,k,m}$. Hence it is enough, for the computations, to keep only the terms with $k > 0$ or, if $k = 0$, with $m \geq 0$,

3. For x and y one should have $j \equiv 0 \pmod 2$, and for z one should have $j \equiv 1 \pmod 2$.

4. The Lindstedt–Poncaré method can be normalized in such a way that for $(k, m) = (1, 0)$ the only term appearing in x corresponds to $(i, j) = (1, 0)$. In an analogous way if $(k, m) = (0, 1)$ the only term in z corresponds to $(i, j) = (0, 1)$.

5. In d and f there are only terms of the form $\alpha^i \beta^j$ with i and j even.

The computation of the terms in (15) is done by increasing order of $n = i + j$. For $n = 1$ one has, from (14), the values

$$x_{1,0,1,0} = 1/2, \quad y_{1,0,1,0} = \overline{k}/2, \quad z_{0,1,0,1} = 1/2.$$

Taking into account the properties 1) to 5) one obtains x_n, y_n, z_n (i.e, the terms of total order n in α and β) and, if n is odd, also d_{n-1} and f_{n-1}. In the determination of x_n, y_n, z_n one should impose the conditions $x_{i,j,1,0} = 0$ if $(i, j) \neq (1, 0)$ and $z_{i,j,0,1} = 0$ if $(i, j) \neq (0, 1)$ (see property 4)). This is what allows to obtain d_{n-1} and f_{n-1} for n odd. Details on the recurrences and on the practical implementation concerning the storing of the coefficients can be found in [34].

Finally we make some a priori comments on the running space and CPU time. The programs have been structured in such a way that we can modify a few parameter instructions by setting

$$NG = (j^4 + 7j^3 + 20j^2 + 26j + 6)/6, \quad NG1 = 2j + 1, \quad NG2 = (j^2 + 3j + 2)/2,$$

if the maximal order to run the program is $n = 2j + 1$. One needs $20 + 2n$ vectors of dimension NG and a few ones of dimensions $NG1$, $NG2$. The total space required (in double precision) is $(4j^5 + 56j^4 + 236j^3 + 691j^2 + 881j)/6 + ctant$. If $n = 35$ this is close to 15.5 Mbyte, and it is close to 48 Mbyte if $n = 45$. To keep in mind a simpler expression, the required space up to order n is proportional to $n^4(n + 23)$ for n large. An analysis of the program shows that the CPU time is roughly proportional to n^8.

3.3 Results and Tests

The program described in the above section, has been run for several values of n, both for L_1 and L_2 and for different values of μ. The results below for $\mu = \mu_{\mathrm{Moon}}$ have been run at order 45. The CPU time on an HP 9000/735 computer is 35^m.

The projections on the (X, Y), (X, Z) and (Y, Z) planes of a 2D torus corresponding to $\alpha = 0.15$, $\beta = 0.40$ appear on figures 4 and 5. The expansions above can be used, if $\beta = 0$ or $\alpha = 0$, respectively, to produce the Lyapunov families of "horizontal" and "vertical" orbits, respectively. The figure 6 shows some of them for several values of α (projection (X, Y)) and several values of β (projections (X, Y) and (Y, Z)). In this figure we have used (9) to display the results in synodical coordinates. We note that the "vertical" orbits project on the (X, Y) plane as a curve travelled twice. The (X, Z) projection, not displayed here, looks like an arc of curve travelled twice. The orbits are, approximately, on a vertical cylinder.

For completeness we also display some "halo orbits". These orbits are born from the planar Lyapunov family when they lose stability inside the center manifold of L_2. That is, beyond the initial hyperbolic couple of variables, common to all the orbits in a vicinity of L_2, another couple becomes unstable. This happens when the vertical mode around the planar Lyapunov orbit becomes resonant with the period of the orbit in a 1 to 1 resonance. They are obtained by an expression like (15) but now both angles, $\theta_s, s = 1, 2$ are equal and the amplitudes (α, β) satisfy some relation $g(\alpha, \beta) = 0$, also derived along the Lindstedt–Poincaré procedure (see [28]). The figure 6 shows the (X, Y), (X, Z) and (Y, Z) projections of these orbits for several values of β. The corresponding values of α follow from the $g(\alpha, \beta) = 0$ relation, but in the range shown for β, the amplitude α changes only between 0.1853 and 0.2318 .

The analytical solutions of (8) have been checked against numerical solutions as follows. Given $\rho > 0$ and $\theta \in [0, \pi/2]$, α and β have been obtained by $\alpha = \rho\cos(\theta)$, $\beta = \rho\sin(\theta)$. Then the phases ϕ_1 and ϕ_2 have been taken as multiples of $\pi/4$. Due to the symmetry it is enough to take $\phi_2 < \pi$. From these data we obtain the initial conditions

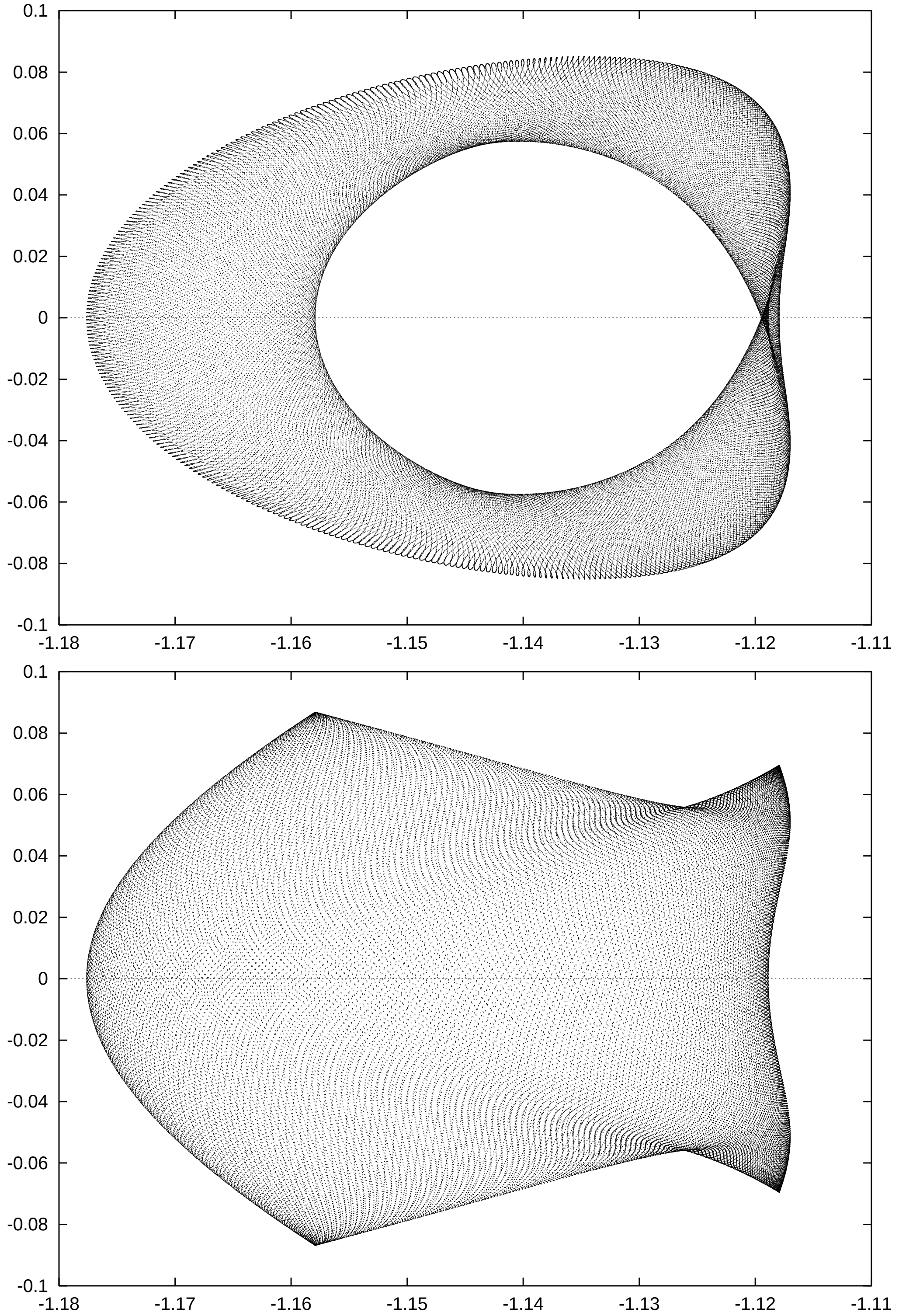

Figure 4: (X, Y) and $(Z, \dot{Z})$ projections of a 2D torus corresponding to $\alpha = 0.15$, $\beta = 0.40$, obtained by the Lindstedt–Poincaré method.

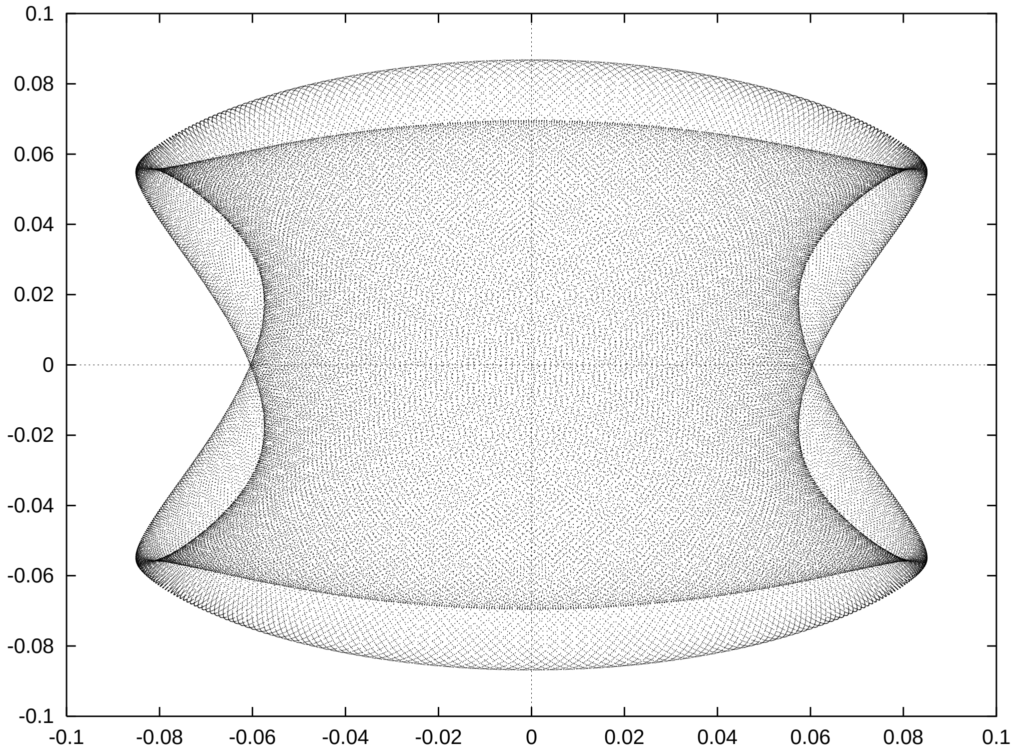

Figure 5: (Y, Z) projection of the previous figure.

for X, Y, Z, $\dot{X}$, $\dot{Y}$, $\dot{Z}$ at $t = 0$, and using (9) the corresponding synodical coordiantes are found. Then, a numerical integration has been started and the distance between the numerical solution and the analytical one is computed (in the phase space) at regular intervals of time up to π. This time is roughly the period of both, the horizontal and the vertical modes. Then, given θ, a value of ρ, ρ_{max}, has been obtained so that for $\rho < \rho_{max}$ the maximum error for all these times and initial phases is less than 10^{-6}. Taking into account that the unstable component increases by a factor greater than 1000 in this time interval, the initial errors are, physically, of the order of a few meters. The values of ρ_{max} versus θ are shown in figure 7. They are given in the (α, β) plane as the inner confining curve.

In the same figure two more curves, confining a region of the (α, β) plane are shown. They correspond to the following. Given θ, it is possible to substitute in the expressions of d and f as functions of ρ. Only even terms appear. It is possible, to estimate, in different ways, a "practical radius of convergence" for the expansions up to order $n = 45$ (despite the series *do not converge*). One method is to look for a behaviour of the form $A \times B^m$ for the coefficient of degree m in ρ, to fit in logarithmic scale, to derive values of B for both d and f, to take the maximum and to use the inverse as an estimate of the radius of convergence. This gives the outer curve in figure 7. Another possibility is to ask for $|t_{44}| < 10^{-3}$, where t_{44} denotes the largest of the last terms in d and f for a given value of ρ. This gives the intermediate curve between the previous two.

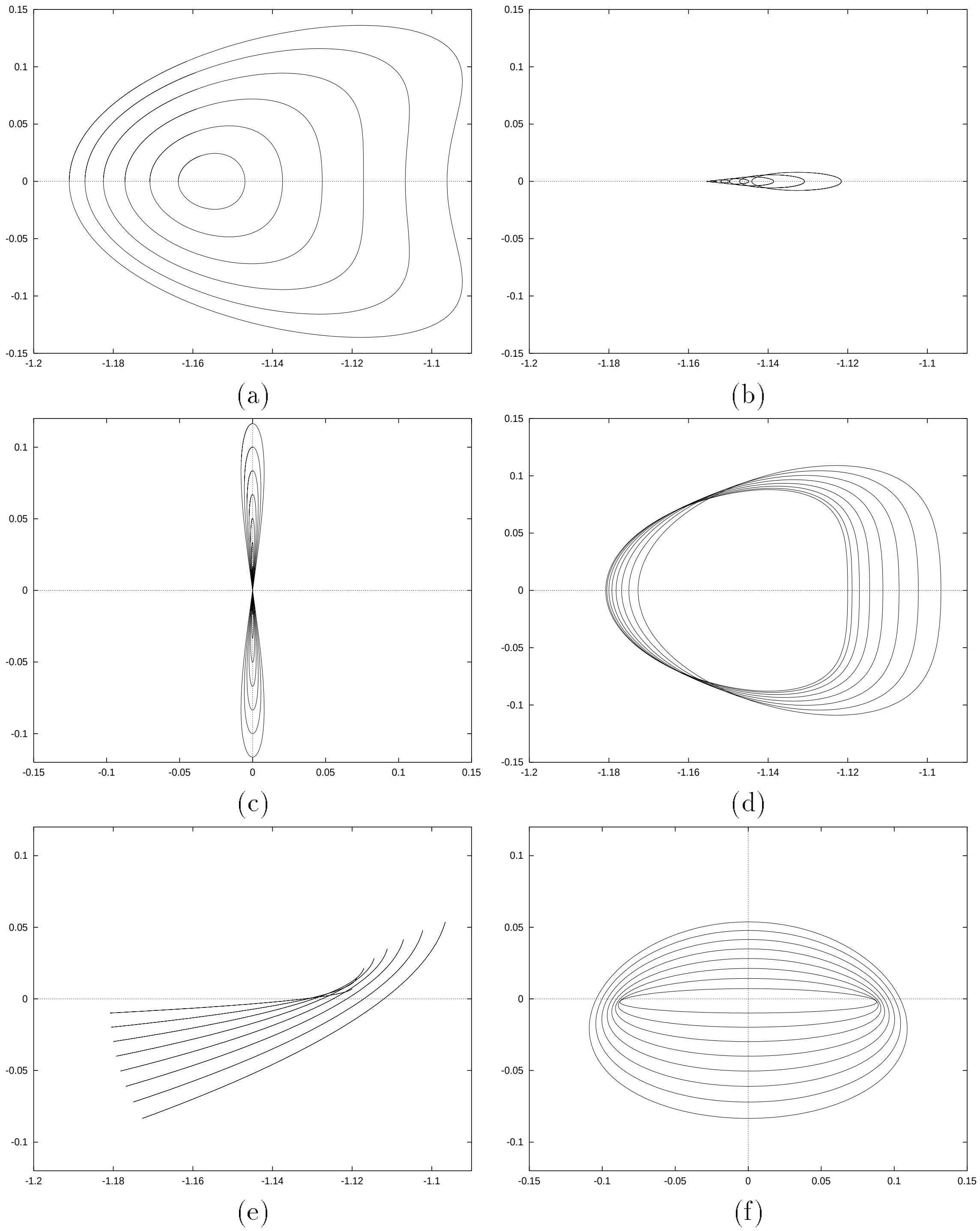

Figure 6: Several projections of periodic orbits obtained using the representation (8): (a) (X, Y) projections of some planar orbits for $\alpha = 0.05, 0.1, 0.15, 0.2, 0.25, 0.3$. (b) and (c) display the (X, Y) and (Y, Z) projections, respectively, for the values of $\beta = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7$. (d),(e) and (f) show the (X, Y), (X, Z) and (Y, Z) projections of orbits of the halo family for $\beta = 0.05, 0.1, 0.15, 0.2, 0.25, 0.3, 0.35, 0.4$.

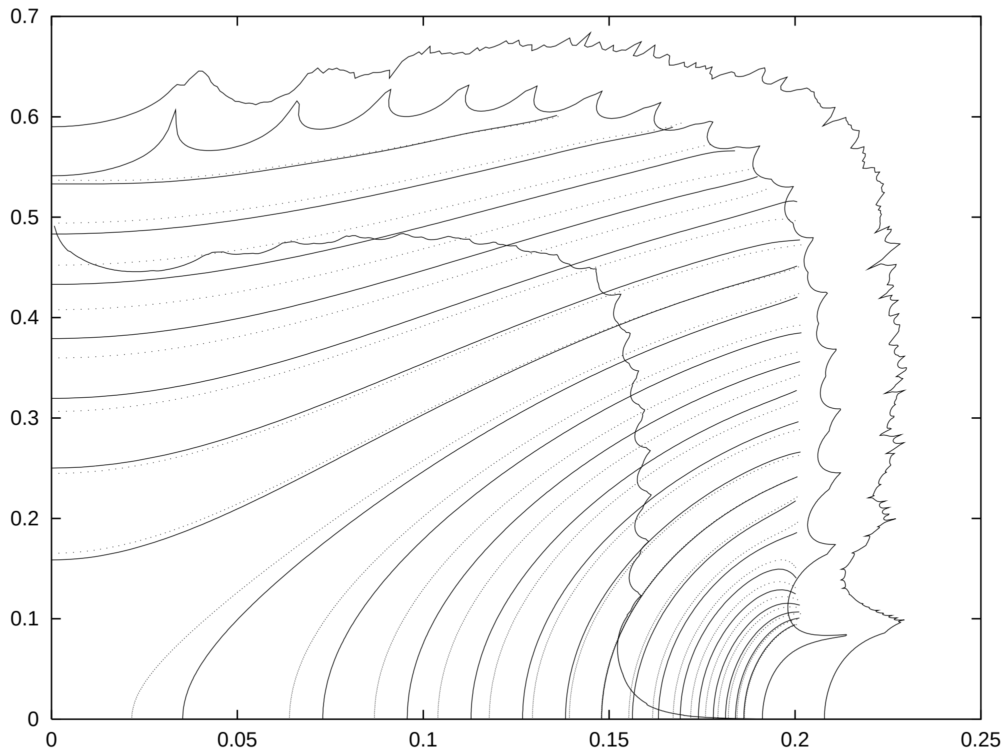

Figure 7: In the (α, β) plane several curves are shown. The three curves going from the α to the β axes define several estimates of the region of "practical stability". The inner one produces very good results when checked against direct numerical results. The two families of curves going from the right lower corner to the left upper one correspond to constant values of difference (continuous curves) and quotient (dotted curves) of the frequencies. See additional explanations in the text.

In the same figure appear two more families of curves. They correspond to constant values of the difference and the quotient of d and f. The values displayed for the difference range from 0.005 to 0.110 with stepsize 0.005 . When the difference increases the curves move from the right lower corner to the left upper one. For the quotient the range goes from 1.0025 to 1.0625 with stepsize 0.0025 .

Further considerations and the applicability of all these families of orbits to spacecraft missions can be found in the references [28], [31],[34] and [38].

3.4 The reduction to the central manifold around L_2

We want to understand how the orbits computed in the previous subsections are organized, to have a global description of a neighborhood of L_2.

First of all we think about skipping the unpleasant unstable terms of the Hamiltonian. This is accomplished by the reduction to the central manifold W^c of dimension 4. We describe how to carry out the computations. After perfoming a linear change of variables going from x, y, p_x, p_y to x_1, x_2, y_1, y_2 and putting $x_3 = z$, $y_3 = p_z$, we have that the Hamiltonian K given by (13) can be written as

$$M = \lambda x_1 y_1 + \frac{1}{2}\omega_1(x_2^2 + y_2^2) + \frac{1}{2}\omega_2(x_3^2 + y_3^2) + \sum_{j\geq 3} M_j(x_1, x_2, x_3, y_1, y_2, y_3),$$

where M_j denotes a homogeneous polynomial of degree j.

Then we pass to an intermediate normal form just trying to cancel in M all the terms such that the total degree in the (x_1, y_1) variables is 1. This is accomplished by making a canonical transformation with a suitable generating function (to be determined). This is possible because there are not problems of small divisors. The denominators appearing in the determination of the generating function (using some complexification to reduce the amount of computation) are of the form

$$(k_1 - l_1)\lambda + \sqrt{-1}(k_2 - l_2)\omega_1 + (k_3 - l_3)\omega_2,$$

with modulus bounded from below by $|\lambda|$. Let $\widetilde{M}$ be this intermediate normal form Hamiltonian. By rearranging terms it can be written as

$$\widetilde{M} = M^0(x_2, x_3, y_2, y_3) + \sum_{j_1+j_2>1} (x_1^{j_1} y_1^{j_2}) M^{j_1,j_2}(x_2, x_3, y_2, y_3).$$

One should have in mind that the new variables are not the sames as the previous ones, despite we keep the same name. The transformation is close to the identity, the closer as the closer is the orbit to L_2.

The fact that small divisors problems do not show up do not means at all that the procedure is convergent. In general one obtains only a $\mathcal{C}^\infty$ manifold. The norm of the coefficients of order n increases as $n!$.

It is immediately seen that $I_1 = x_1 y_1$ is a first integral. The reduction to the center manifold is obtained by setting $I_1 = 0$. So it remains a two degrees of freedom Hamiltonian, M^0, and furthermore

$$M^0 \quad = \quad \frac{1}{2}\omega_1(x_2^2 + y_2^2) + \frac{1}{2}\omega_2(x_3^2 + y_3^2) + \sum_{j\geq 3} M_j^0(x_2, x_3, y_2, y_3). \tag{16}$$

We point out that if in (16) we put $x_3 = y_3 = 0$ we have an invariant set under the flow associated to M^0. This is now a 1 degree of freedom Hamiltonian. The orbits are simply the Lyapunov periodic orbits around L_1.

By keeping $M^0 = h$ fixed we can use $x_3 = 0$ with $y_3 > 0$ as surface of section. This section, Σ, is not global because it fully contains the corresponding Lyapunov orbit. But we can think that all the points of this orbit (in the boundary of the section) are identified. So we should obtain an S^2 sphere (the classical Hopf fibration) that can be seen as an space of "osculating orbits". For some value of h, h_H say, the level $M^0 = h$ contains the Lyapunov orbit bifurcating to halo orbits. From this value on, the section Σ contains two fixed points associated to the two (symmetrical) halo orbits. It is known that, at least for moderate values of h, they are of elliptic type. At least for values of h not too far from h_H we have in S^2 four fixed points. Two of them correspond to the Lyapunov orbit (hyperbolic) and to the almost vertical periodic orbit (elliptic), respectively. The other two are the halo orbits (elliptic).

A program carrying out this reduction has been developed, implemented and run (up to total order 16). Let ξ and η denote the new variables after skipping the hyperbolic terms in the intermediate normal form. When $M^0(\xi, \eta)$ is available we can do several simulations. The figure 8 shows the result of these simulations for a value of the energy such that the halo orbits are clearly visible. This corresponds to a Poincaré section through $\xi_2 = 0$ (close to $z = 0$ in the initial coordinates). Near the center of the figure one can see a fixed point. It corresponds to a "vertical" periodic orbit. It is surrounded by invariant tori. They belong to the family computed at the beginning of this section, despite the amplitudes are too large and can not be well approximated by the results of the Lindstedt–Poincaré method. The external curve is the Lyapunov planar orbit sitting on this level of the energy. Two other fixed points correspond to the two halo orbits, which are symmetrical the one from the other with respect to $z = 0$) . They are, in turn, surrounded by invariant 2D tori. Between the 2D tori around the vertical orbit and the ones around the halo orbits there is the trace of the stable and unstable manifold of the planar Lyapunov orbit. However, despite the energy is not close to the one of the L_2 point, even the stochastic zone associated to these manifolds is hard to see.

We note that the figure 8 has been displayed with a scale different from the one used in the figure 6. The variables are the ones corresponding to the intermediate normal form. The Moon is located outside the figure, in the negative vertical axis of figure 8.

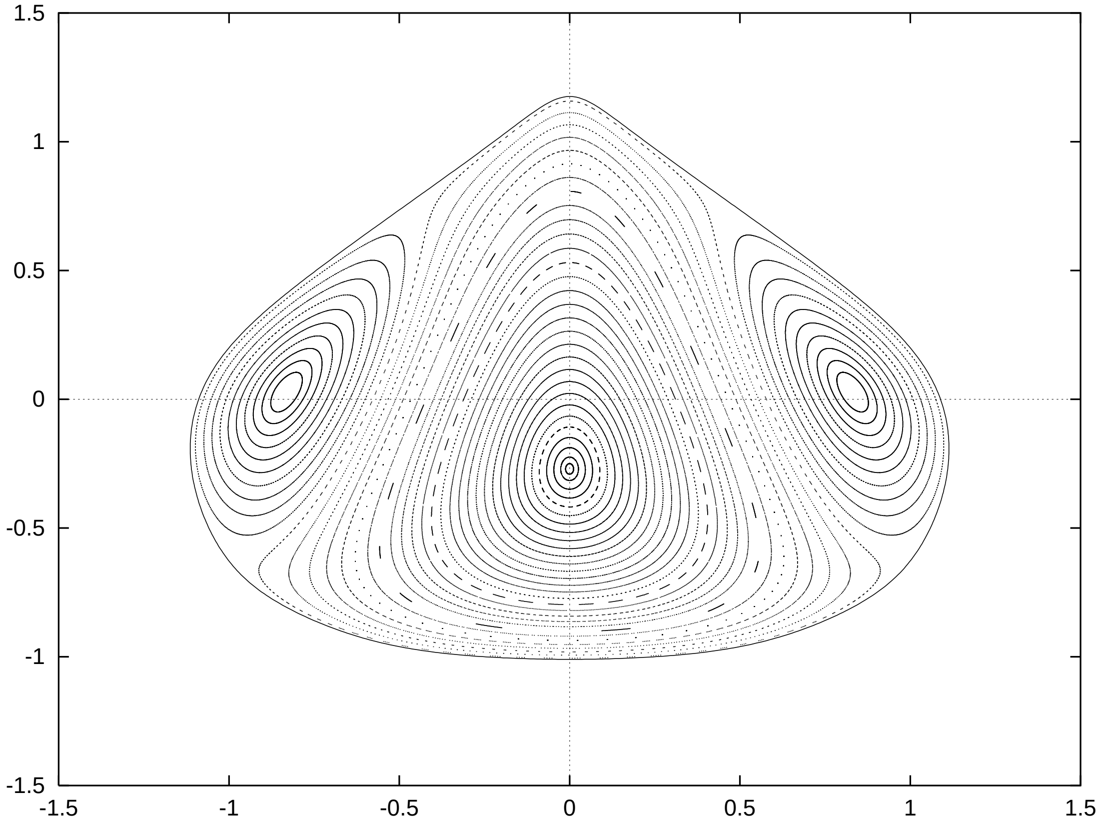

Figure 8: A Poincaré section of the flow of the Hamiltonian reduced to the central manifold of the L_2 point in the Earth-Moon case, for a given level of the energy. See additional explanations in the text.

4 Orbits Near the L_5 Point in the RTBP model. Practical Stability Region Boundaries and Numerical Computation of Unstable Invariant Tori

The RTBP has also libration points sitting outside the line containing the massive bodies. These are the well known Lagrange or triangular points. Using the notation of the previous section they are at rest, in the synodical system, at $(\mu - 1/2, \pm 3^{1/2}/2, 0)$. All these libration points (both collinear and triangular) appear also in the general three body problem.

One of the classical problems in Celestial Mechanics concerns the stability of the triangular points. We collect here some results.

- They are linearly stable for the mass parameter $\mu \in [0, \mu_1]$, where

  $$\mu_s = \frac{\omega_{\text{short}}}{\omega_{\text{long}}} = s, \quad s \in \mathbb{N},$$

  and ω_{short} and ω_{long} are the frequencies at $L_{4,5}$ in the planar case [98]. Their values are

  $$\left[\frac{1}{2}(1 \pm (1 - 27\mu(1-\mu))^{1/2})\right]^{1/2}.$$

  Hence $\mu_1 = (1 - (23/27)^{1/2})/2 \approx 0.038521$.

- The problem has nonlinear stability for $\mu \in [0, \mu_1] \setminus \{\mu_2, \mu_3\}$ in the planar case [65].

- The nonlinear stability of the spatial problem is unknown. Probably there is some amount of Arnol'd diffusion.

- There is a domain of "practical stability" which is not too small. We summarize how this result is obtained [27], [88].

  Let $H = H_2 + H_3 + \ldots$ be the power expansion of the Hamiltonian around $L_{4,5}$, where H_k contains the homogeneous terms of degree k in positions and momenta. By a linear transformation, H_2 can be reduced to

  $$-\frac{1}{2}\omega_s(q_1^2 + p_1^2) + \frac{1}{2}\omega_l(q_2^2 + p_2^2) + \frac{1}{2}\omega_z(q_3^2 + p_3^2),$$

  where ω_s, ω_l stand for ω_{short}, ω_{long}, and ω_z is the frequency in the vertical direction ($\omega_z = 1$).

  A canonical transformation $(q, p) \longrightarrow (Q, P)$, obtained by means of a generating function $G = G_3 + G_4 + \cdots + G_n$, is applied to put H in normal form up to order n. We have the new Hamiltonian

  $$\mathcal{H} = N_2 + N_3 + \cdots + N_n + \mathcal{R}_{n+1},$$

where N_k are terms of degree k in the normal form and $N_2 = H_2$. If we only keep $N_2 + N_3 + \cdots + N_n$, the system is integrable. Let

$$I_j = \frac{1}{2}(Q_j^2 + P_j^2),$$

the new momenta. Then, the diffusion of the momenta is due to

$$\dot{I}_j = \{I_j, R_{n+1}\},$$

where $\{\cdot,\cdot\}$ stands for the Poisson bracket. As the remainder $\mathcal{R}_{n+1}$ is of the form $R_{n+1} + R_{n+2} + \cdots$ we introduce some norm $|| \ R_k \ ||$. Then one obtains successively bounds for $|| \ H_k \ ||$, $k \geq 2$, $|| \ G_k \ ||$, $2 < k \leq n$ (only a finite number of small divisors appears) and for $|| \ R_k \ ||$, $k > n$. The bounds of $|| \ R_k \ ||$ are given by means of a recurrence which depends only on the norm of the homogeneous parts of the initial Hamiltonian, $|| \ H_k \ ||$, and on the current small divisors which appear till order n.

In a ball of radius ρ in the (Q, P) variables one has

$$| \ \mathcal{R}_{n+1} \ | < \sum_{k>n} \rho^k ||R_k||,$$

where $| \ \ |$ denotes the sup norm. In a similar way we can bound $|\dot{I}_j|$, the speed of diffusion.

Given T, δ, there exists an initial radius, ρ_0, such that if $(Q, P)_{t=0} \in B_{\rho_0}$ then $(Q, P)_t \in B_{\rho_0(1+\delta)}$ for all $|t| < T$, where B_ρ denotes the ball of radius ρ centered at the origin. We remark that one obtains better results if H_k, N_k, G_k are computed explicitely up to some order for the desired value of μ by means of a symbolic manipulator.

As a result one obtains, in general, Nekhorosev type estimates (i.e., for δ fixed, one has $T \approx \exp(c/\rho^d)$, for some positive constants c, d).

- Numerical simulations ([34], [38]) show a "stable domain" even larger than the one previously described for the planar case [64].

Partial results are known for the elliptic restricted three body problem (i.e., the massive bodies move on elliptic orbits), both concerning local analysis, "practical stability" ([50]) and numerical determination of the "stable domain" ([20], [38]). For the real problems the situation can be more involved. For instance, in the case of the triangular points of the Earth–Moon system, the most important deviation from the RTBP model is due to the influence of the Sun. It has been established that these points are unstable, but regions of practical stability (at least for time intervals very long compared to the duration of space missions) exist at some distance from the triangular points, ranging from 1/4 to 3/4 of the Earth–Moon distance ([38], [96]).

In this section we shall concentrate on the RTBP model for $\mu = 0.0002$. There are several reasons for this choice. From one side it is small enough to consider that perturbation theory can give good approximations. From the other it is big enough so that one needs not a very big amount of time to detect some escapes. Finally it is also close to the actual mass ratio in the Saturn–Titan system.

4.1 Boundaries of the Practical Stability Region

Near the triangular points the system can be considered as three harmonic oscillators, with frequencies at the point equal to ω_{short}, ω_{long} and ω_{vert}, this one being equal to 1. For μ small the values of the ω_{short}, ω_{long} are, respectively, $1 - 27\mu/8 + O(\mu^2)$ and $(27\mu/4)^{1/2} + O(\mu^{3/2})$. Hence, one can also consider, as basic in a neighborhood of L_5, the frequencies ω_{vert}, ω_{long} and $\omega_{\text{vert}} - \omega_{\text{short}}$, whose values are of the order of 1, $\mu^{1/2}$ and μ, respectively.

Provided the frequencies satisfy some conditions (not too close to a strong resonance) there is a "practical stability" region. The previous consideration on the orders of magnitude shows that no strong resonance occurs if μ is small, except the 1 to 1 resonance between the short period and vertical modes. Hence, it takes an extremely large time span to increase the actions by a significant amount. We are interested in the global shape of these regions, because the results of [27] and [88] give rather pessimistic estimates.

To this end the following experiment has been done. Given values of ρ, α and Z, we consider an initial point with synodical coordinates

$$X = \mu + (1+\rho)\cos(2\pi\alpha), \qquad Y = (1+\rho)\sin(2\pi\alpha), \qquad Z,$$

and with zero initial velocity: $\dot{X} = 0, \dot{Y} = 0, \dot{Z} = 0$. The L_5 point corresponds to $\rho = 0$, $\alpha = 1/3$ and $Z = 0$. Then a numerical integration is started up to a given final time. Points close to L_5 just remain moving around it (typically on a 3D torus). But if the initial conditions are sufficiently far from the libration point, the projection of the orbit on the (X, Y) plane crosses the X axis and then it can be considered as escaped from a "large" vicinity of L_5. Due to the shape of some large tori which seem to exist even for large values of Z, it is convenient to take as "escape" criterion from the vicinity of L_5 the condition $Y(t) < Y^*$ for some negative value of Y^*. As a practical value one has taken $Y^* = -0.5$. For the current value of μ we have found orbits lying on tori "around" L_5 such that $Y(t)$ reaches values up to -0.233 . The time span used for the computations is 10,000 times the period of the massive bodies. In the case Saturn–Titan, this means more than 400 years. This can seem a short time interval, but see the comments below.

The figure 9 displays part of the results of this experiment. It gives the projection on the (X, Y) plane of the points which do not escape, according to the previous criterion, starting at the planes of Z value equal to 0.0, 0.3, 0.4, 0.5, 0.6, 0.7 and 0.8 . The ones

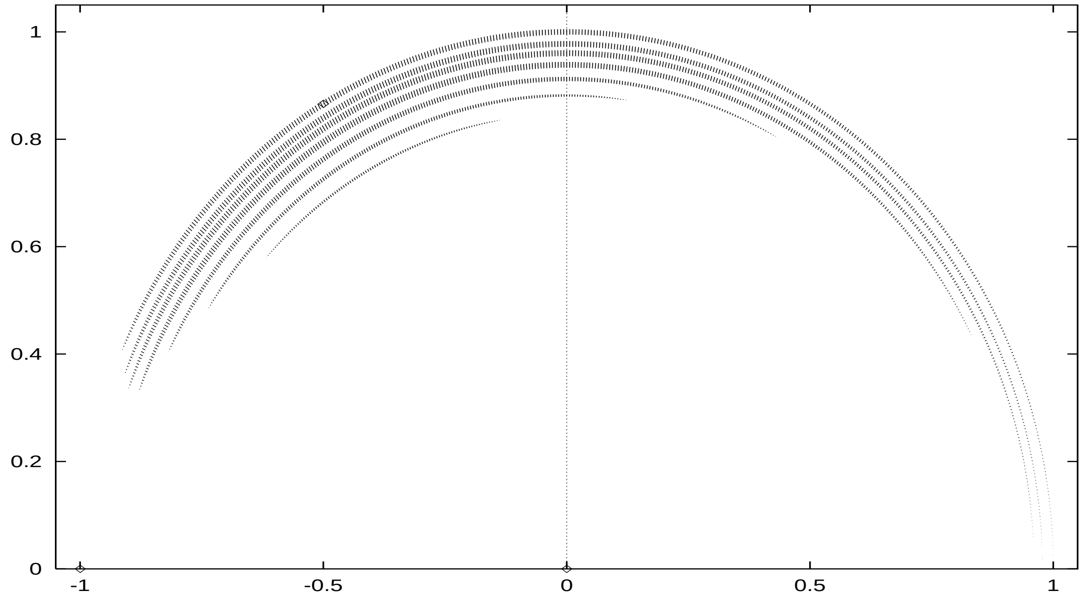

Figure 9: Projections of some sections of the global stability region around L_5 for $\mu = 0.0002$. See additional explanations in the text.

with initial Z equal to 0.1 and 0.2 overlap the projection of the case $Z = 0.0$. The stepsizes in ρ and α have been taken equal to 10^{-5} and 10^{-3}, respectively.

Some remarks must be made concerning the results.

- For small values of μ, as in the present case, the transition from the "stable" zone to instability is quite sharp. This means that increasing the 10,000 revolutions to 100,000, for instance, produces very few losses. On the other hand, most of the points which escape go away from the vicinity of L_5 in a few revolutions. Most of the points which escape after a longer time go away after a time interval of the order of $1/\mu$. This is related to the smallest of the basic frequencies, $\omega_{\text{vert}} - \omega_{\text{short}}$, as said before.

- For the planar case it is seen that the nonescaping points fill an area that we want to identify. To this end take the *zero velocity curve* (z.v.c.) passing through the collinear point L_3, located beyond the larger body at a distance rather close to the distance between the massive bodies. The z.v.c. corresponds to points (X, Y) at which the massless particle at rest has a given value of the energy, in our case the same energy that at L_3. The upper part of this curve (on $Y > 0$) goes, in the angular variable, from $\alpha = 0$ up to a value which tends to $\alpha_{lim} = \cos^{-1}(-(2^{1/2} - 0.5))/\pi \approx 156.09$ degrees, when μ tends to zero. Consider the curve with $\rho = 0$ (that is, at unit distance from the larger body). Then the z.v.c. is located at a distance $O(\mu^{1/2})$ from it. More concretely, there is a function $K(\alpha)$ having a double zero at $\alpha = 0$ and a simple one at $\alpha = \alpha_{lim}$ such that the z.v.c.

is described by $\rho = \pm(K(\alpha)\mu)^{1/2} + O(\mu)$. The "boundary" of the stability region on $Z = 0$ seems to be $\rho = \pm\frac{1}{2}(K(\alpha)\mu)^{1/2} + O(\mu)$.

- There is numerical evidence that, in the planar case, the boundary of the "stability" region is related to the stable and unstable manifolds of the center–stable manifold at L_3 (see [38]). This is a codimension 1 manifold. In particular it contains the stable and unstable manifolds of the family of planar Lyapunov orbits around L_3.

- The full set of points in the "stability" zone with zero synodical velocity, in the spatial RTBP, seems to be on a thin shell. The shape of the shell is essentially circular in the horizontal direction and parabolic in the vertical one. Cutting the "stable" zone by $\alpha = 1/3$, the central point, as a function of Z, is close to $\rho = -0.245Z^2$.

- There are families of unstable 2D tori which play also a role in the boundary of the "stability" region, as we shall show in the next subsection. If the initial value of Z is small these tori reach a vicinity of the L_3 point. Their sections through $Z = 0$ have positive X values. For large initial values of Z, the sections through $Z = 0$ of the 2D tori which seem to be at the "boundary" of the stable region have negative X values.

Summarizing, according to the numerical experiments in the planar case, most of the stable motions take place in tori 2D and they are "bounded" by planar periodic orbits of the Lyapunov family around L_3. In the spatial case, most of the stable motions take place in tori 3D and they are "bounded" by 2D tori associated to the 1 to 1 resonance between the short period and the vertical frequencies. These 2D tori are located at different places depending, mainly, on the initial value of Z. Of course, high order resonances play a role and give rise also to chains of heteroclinic connections taking points away, probably from any neighborhood of any stable point, but this requires a very large amount of time and this mechanism can only be seen for rather large values of μ (say, of the order of 10^{-2}) .

4.2 Unstable 2D Tori: Detection and numerical computation

The figure 10 shows the first 100,000 iterates of the Poincaré section through $Z = 0$ of points starting at rather close points. The first one has been computed with $\rho = -0.1506066340$, $\alpha = 1/3$, $Z = 0.8$ and zero synodical velocity. The second one differs only in the last digit of ρ, which has the value -0.1506066339 . They show, not only the sensitive dependence with respect to the initial conditions, but also the existence of a 2D torus which separates both kinds of motion.

To make last assertion more clear figure 11 shows the first iterates of the Poincaré map, starting at both initial conditions, on the same plot. After passing close to a

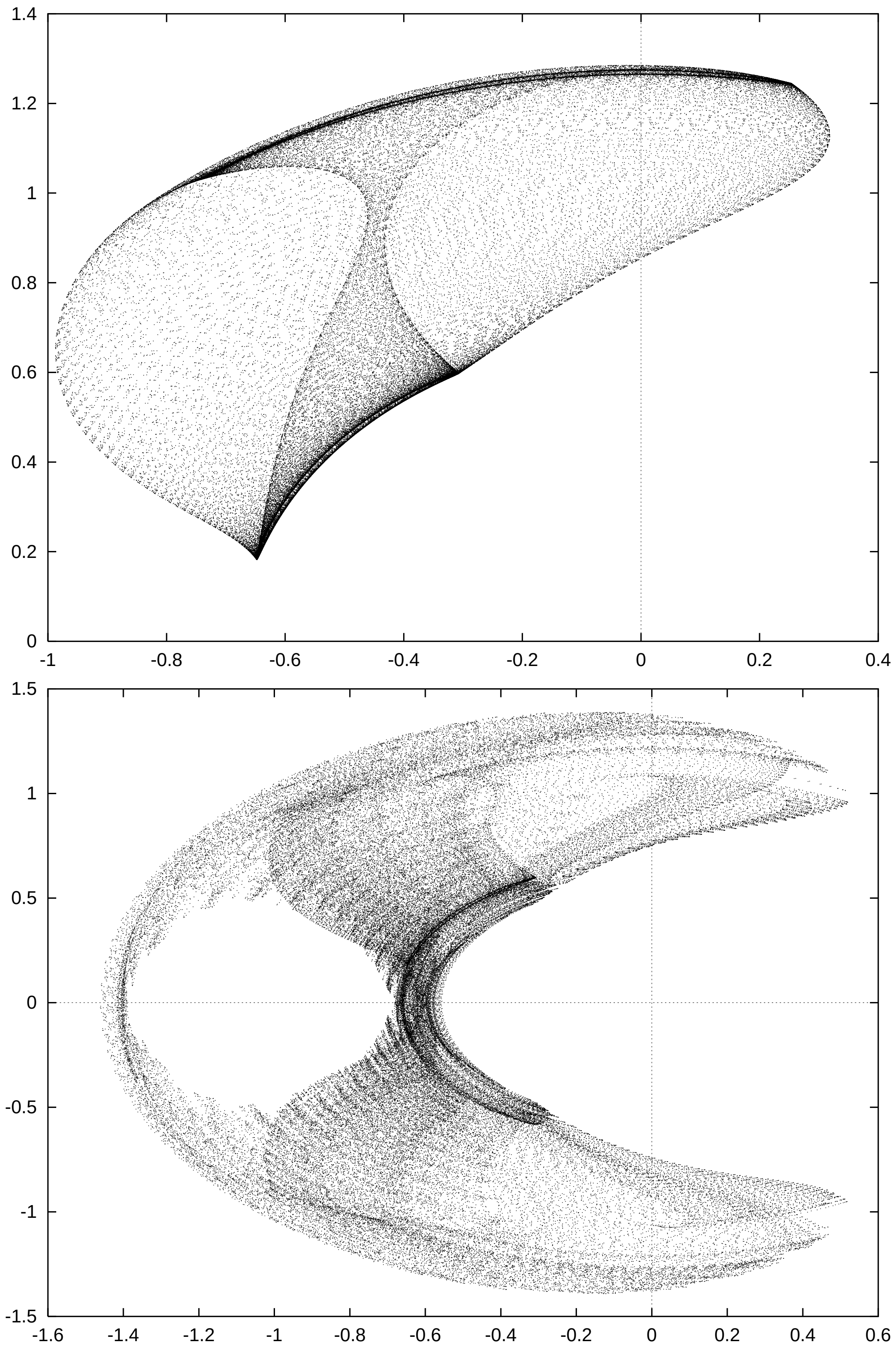

Figure 10: The first 100,000 iterates of the Poincaré map through $Z = 0$ starting with zero synodical velocity at $\alpha = 1/3$, $Z = 0.8$. Top: initial $\rho = -0.1506066340$. Bottom: initial $\rho = -0.1506066339$.

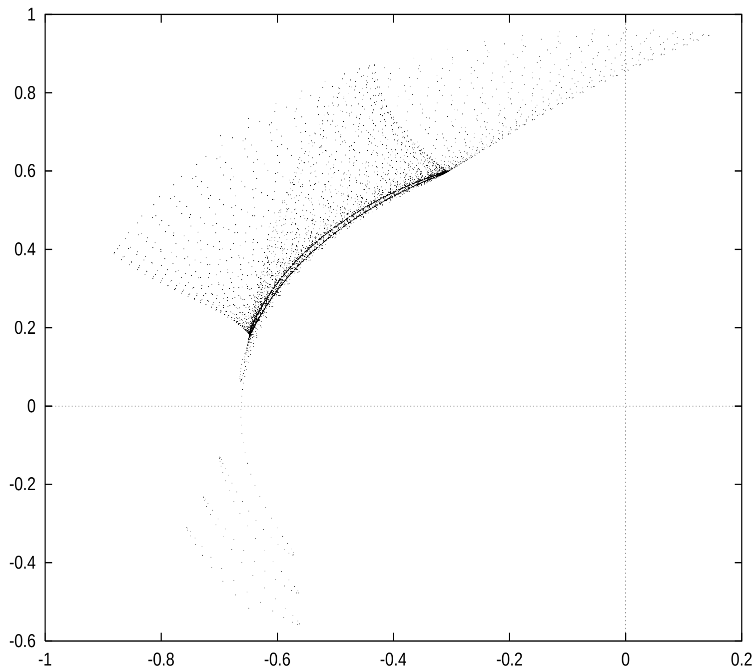

Figure 11: The first 5,000 iterates of the Poincaré map for a point on a 3D torus and the first 4,165 iterates of an escaping point. See the initial data on the caption of figure 7 . Both remain for near 2,000 iterates close to an invariant curve of the Poincaré map.

hyperbolic invariant curve (the intersection of the 2D torus with $Z = 0$), the orbits separate in an exponential way. It is worth to mention that the motion of the point not lying on a 3D torus approaches, in the sequel, several other 2D tori. Looking at suitable time intervals it seems to be rather close to some 3D tori, but this one changes (usually in a fast way) from time to time. This is an evidence of the homoclinic/heteroclinic tangle of the spatial RTBP.

We are interested now in the computation of this unstable 2D torus. To this end we fix an energy and use the Poincaré section through $Z = 0$. In this way the problem reduces to compute an invariant hyperbolic curve of a 4D symplectic transformation. As there are, in general, cantorian 1–parameter families of such objects, we should also fix some rotation number. To select some values of energy and rotation number we proceed as follows. The energy has been choosen (in some arbitrary way) as the average of the energies of the two previous orbits. To have some rotation number, first, we look at the iterates of the two previous orbits which remain close to the desired curve. It is possible to obtain the successive maxima and minima in any one of the current variables ($X, Y, \dot{X}, \dot{Y}$, the value of $\dot{Z}$ being derived from the energy relation (12)) by using interpolation of the different data versus number of iteration. This gives a rotation number $\nu \approx 0.02189307$, that we shall use in the sequel. It is also important the inverse of this number $N = 1/\nu \approx 45.67656$.

To compute a periodic orbit (either stable or not) for a map is, in theory, a simple problem. Assume we denote the map as T and we look for a periodic point of minimal period k. If p is a point in the orbit, we set the equation $T^k p - p = 0$ and we try to solve it by using, for instance, Newton's method. To have success we need some conditions on the properties of the orbit and the initial data should be close enough to p. Furthermore some technical difficulties can occur, for instance if some of the eigenvalues of $D(T^k)(p)$ is very large. In that case a common and very efficient procedure, to solve this difficulty, is based on *parallel shooting* (see, for instance, [97] and also [28]).

The problem is different in the case of an invariant curve. Of course, one can use expansions, as we did in the previous section, if we have some information on a nearby problem, or if the curves are close to some known point. Another possibility is to look for a representation of the coordinates of the curve as a (truncated) Fourier series and to set up a system of equations for the coefficients. This is efficient when the expression of the map is available explicitely. In the present case we want to proceed by direct numerical methods.

As we do not have a return map, as in the case of the periodic orbit, it is possible to *synthetize* it. We look for a fixed point $\mathcal{R}(p) = p$ of the return map $\mathcal{R}$, where $\mathcal{R} = T^N$ with $N = 45.67656$. Certainly we can not compute the power 45.67656 of the Poincaré map, but it is possible to compute the powers in a range $[N_1, N_2]$ containing the desired value of N and compute $\mathcal{R}$, as well as its differential, by interpolation. However the numerical problem has not an unique solution. Indeed, all the points on the curve must be fixed points of the synthesized return map. To determine an unique solution we can fix some of the coordinates of the point p.

In the present problem we have fixed the value of $Y = Y^* = 0.3897977$. This is an arbitrary choice because any choice not too close to the extrema values of Y on the curve could be used, The value that has been taken corresponds to one of the iterates of the Poincaré map for the point sitting on a 3D torus. This iterate will be taken as starting point of the Newton method. Only 3 coordinates $(X, \dot{X}, \dot{Y})$ remain now free, and they are obtained successfully. Note that the Y coordinate of the return map is not asked to coincide with Y^*. This should happen automatically, and is being used as a check of the computations. The Newton process is stopped when the return errors are below 10^{-13}. This is attained in 4 iterates of the procedure. The check of the initial and final values of Y gives also a difference below 10^{-13}. The results are shown in figure 12.

The differential of the return map $D\mathcal{R}$ also provides an estimate of the hyperbolicity of the invariant curve. The dominant eigenvalue is $\lambda_{max} \approx 1.24568$, which implies a dominant eigenvalue for the Poincaré map along the curve $\lambda_{map} \approx 1.004821$. This is a very mild instability which makes the numerical computations much simpler.

We want to remark that it is possible to use a continuation procedure to compute other invariant curves, either by changing the energy, the rotation number or both. When the rotation number takes a rational value (and in the computer approximation

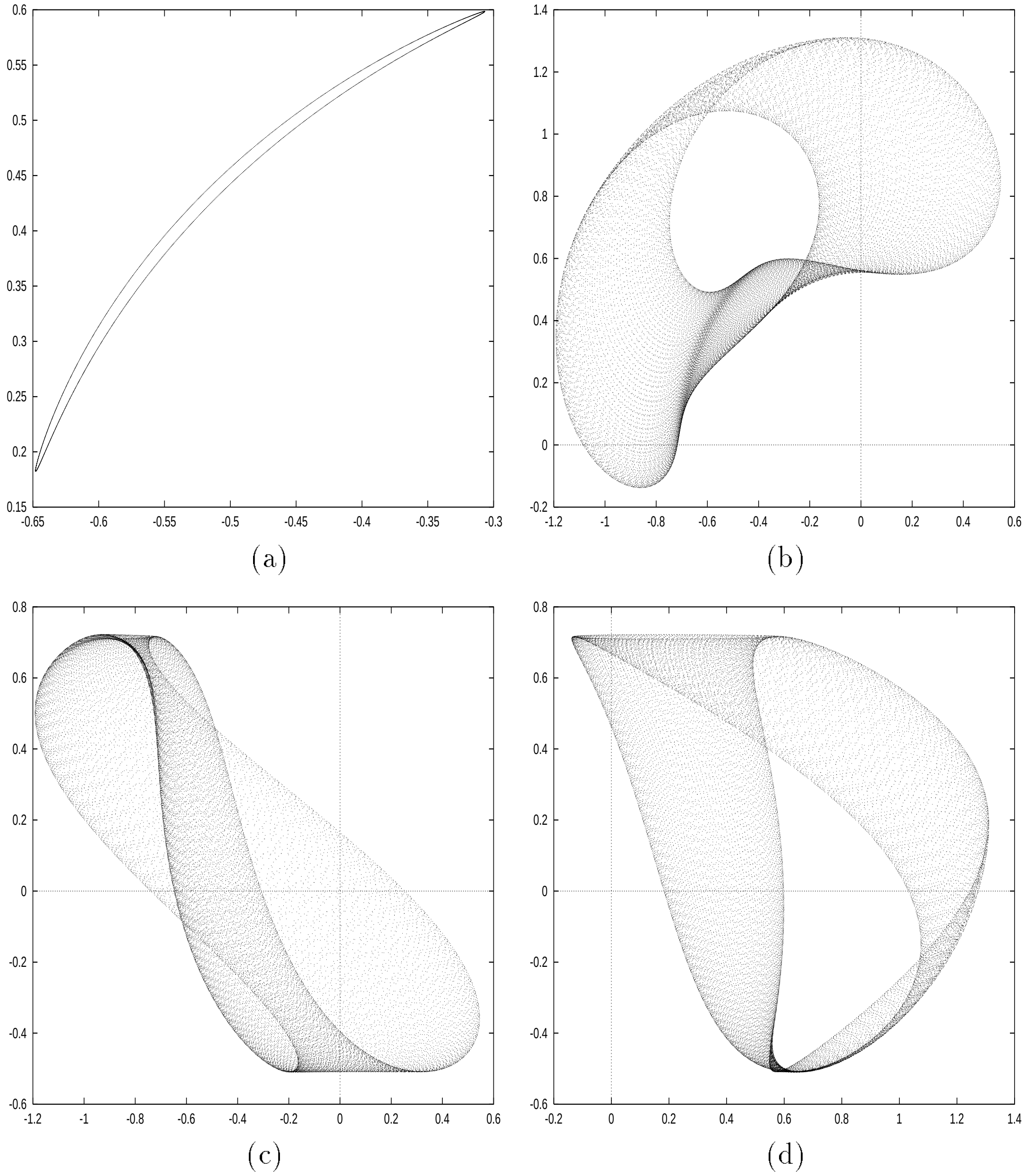

Figure 12: An invariant hyperbolic curve for the Poincaré map through $Z = 0$ of the RTBP as is seen in figure 8 . (a) the (X, Y) projection of the curve. (b), (c) and (d) display the (X, Y), (X, Z) and (Y, Z) of points on the corresponding 2D torus.

this is always the case!), the invariant curve generically do not exists and it is probably replaced by a couple of periodic orbits of elliptic–hyperbolic and hyperbolic–hyperbolic type. Hence, there is no reason at all that any point of these periodic orbits be located on the hyperplane $Y = Y^*$. However, the numerical method can still produce some result without any problem. I tried this both with rotation number $\nu = 1/45$ and $\nu = 1/46$ without trouble. The reason is, obviously, that the return map (T^{45} or T^{46} in this case) has a so slow dynamics that it is below the tolerance of the method (10^{-13}). To see that the method fails in this case it is enough to do all the computations with a higher accuracy.

4.3 A Generalization: Numerical Computation of invariant Tori

The procedure described in the previous subsection can be generalized to the direct numerical computation of invariant tori of flows and maps under reasonable conditions. For simplicity it is sketched here in the case of maps. The case of flows can be reduced to this one by using a Poincaré section. It is important to mention that only a local section is needed.

First we reformulate the procedure above. Let $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$ be the standard 1D torus which is considered as parametrized by values ranging in $[-0.5, 0.5)$. The map on the invariant curve is conjugated to the map $x \longrightarrow x + \nu$ and, hence, after k iterates the current value of x is $x + k\nu \pmod 1$. By taking several values of k (in the concrete example above we have taken k from 38 to 54), one obtains several values of x around 0. By interpolation it is possible to compute the return map $\mathcal{R}$.

In the general case, we look for an m-dimensional torus where the dynamics is conjugated to the map $x \longrightarrow x + \nu$ of the standard torus $\mathbb{T}^m = \mathbb{R}^m/\mathbb{Z}^m$, ν being now in $\mathbb{R}^m$ and satisfying nonresonance conditions: $(k, \nu) \in \mathbb{Z}$ for some $k \in \mathbb{Z}^m$ implies $k = 0$, where $(,)$ denotes the scalar product in $\mathbb{R}^m$. We assume that all the constrains due to first integrals have been taken away and we consider directly the reduced problem. As we are interested on an m-dimensional torus we fix m coordinates of the point p we are looking for. Furthermore we assume that the total dimension n is at least $2m$ and that a variation of the additional $n - m$ coordinates produces a variation of the frequencies ν, provided they exist (probably they do only exist on some cantorian set), which is not degenerate, in the sense that the differential of the frequencies with respect the additional variables has maximal rank. It is also supposed that the remaining $n - 2m$ coordinates, if any, are associated to eigenvalues different from 1. Hence we are assuming, implicitely, that the dynamics along the torus is reducible to constant coefficients (see [49] and [51] in this direction).

Then one considers different iterates of T with number of iterations $k_1, k_2, ..., k_q$ such that on the standard torus the related values of x ($k_1\nu, ..., k_q\nu \pmod 1$) fall on a prescribed neighborhood of 0. Then, it is possible to synthesize the return map $\mathcal{R}$ as the value at zero of some interpolating function, taking the value $T^{k_j}(p)$ at the point $k_j\nu, j = 1, ..., q$. The condition to be imposed on p is that the $n-m$ free coordinates should satisfy $\mathcal{R}(p) = p$, the remaining m coordinates being used as an additional check. This equation can be solved by Newton's method and the required differential is obtained also by interpolation. Finally it is possible to obtain the eigenvalues of T in the additional $n-2m$ coordinates from the differential of the return map.

This procedure is being tested now in different examples, either conservative or not.

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