Surgery on the Limbs of the Mandelbrot Set

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Introduction

The technique of *quasiconformal surgery* on complex polynomials as a way of constructing certain maps between parameter spaces was first introduced in [BD] to relate a certain cubic parameter space to the 1/2–limb of the Mandelbrot set. Afterwards, surgery was used in [BF] to construct homeomorphisms between some limbs of the Mandelbrot set. Recently, it has been announced in [EY] the existence of products of Mandelbrot sets in the two–dimensional complex parameter space of cubic polynomials, also by means of surgery.

The goal of this paper is to outline the results and proofs published in [BF] and presented at the *Göttingen Workshop on Siegel Disks* on December 1995. They are presented hoping that the reader can realize the important points in the proofs without entering into the many details that are necessary to formalize the surgery process. We assume familiarity with the basic definitions and concepts in complex dynamics, although we give a brief summary of the necessary ones to state the main results. From that point on, we refer to the adapted introduction in [BF] where the main tools are summarized, or to the original sources contained in the bibliography. Finally, Sections 4 and 5 contain some new material on further results (in progress) and some conjectures and questions on related topics.

1 Preliminaries and Main Results

For a complex polynomial f of degree $d > 1$, the point at infinity is always a superattracting fixed point. Since infinity has no other preimage than itself, its basin of attraction is connected. The complement of this basin is called the *filled Julia set of* f, that is,

$$K(f) = \{z \in \mathbb{C} \mid \{f^n(z)\}_n \text{ is bounded}\}.$$

The set $K(f)$ is compact and simply connected by the observations above. Its connectedness though, depends on the orbits of the *critical points* of f, or zeros of f'. Indeed, $K(f)$ is connected if and only if all critical points of f belong to $K(f)$ (see, for example, [Bl]).

We define the *Julia set* of f, denoted by $J(f)$, as the common boundary of $K(f)$ and the basin of attraction of infinity. This definition only works for polynomials. In a general

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setting, the Julia set is the set of points for which the family of iterates does not form a normal family.

If f_a is a family of polynomials as the one above that depends on a parameter $a \in \mathbb{C}$, we can define its *connectedness locus* as the subset of the complex plane consisting of parameter values for which the filled Julia set of the corresponding polynomial is connected, i.e.,

$$C(f_a) = \{a \in \mathbb{C} \mid K(f_a) \text{ is connected}\}.$$

The best known connectedness locus is, by far, the one associated to the family of quadratic polynomials $Q_c(z) = z^2 + c$, called the *Mandelbrot set* (see Figure 1). Since the polynomials Q_c have only one critical point at $\omega = 0$, the Mandelbrot set can be defined as

$$M = C(Q_c) = \{c \in \mathbb{C} \mid \{Q_c^n(0)\}_n \text{ is bounded}\}.$$

The properties and structure of the Mandelbrot set are now almost completely understood

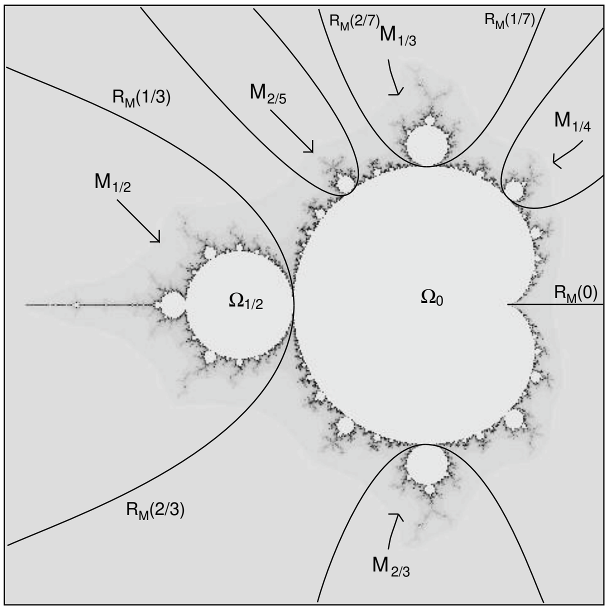

Figure 1: The Mandelbrot set.

thanks to the work of A. Douady and J. Hubbard [DH1, DH2], D. Sullivan [MSS] and J-C. Yocozz among others. However, the question about the local connectivity of M (MLC conjecture) still remains unsolved.

We will also work with other connectedness loci, namely those associated to the one parameter families of polynomials $P_{q,\lambda}(z) = \lambda z(1 + z/q)^q$ (one for each value of $q \in \mathbb{N}$). For a given $q \geq 2$, $P_{q,\lambda}$ has two critical points: $-q$ with multiplicity $q-1$ and $\omega_q = \frac{-q}{q+1}$ with multiplicity one (we will omit the subscript q whenever it creates no confusion). The orbit of

$-q$ is always bounded since $P_{q,\lambda}(-q) = 0$ and 0 is fixed. Hence, the connectedness locus is defined as

$$L_q = C(P_{q,\lambda}) = \{\lambda \in \mathbb{C} \mid \{P^n_{q,\lambda}(\frac{-q}{q+1})\}_n \text{ is bounded}\}.$$

(See Figures 2, 3 and 4). One can easily see [BF, Prop. 3.1] that any polynomial with these properties must be of the form $P_{q,\lambda}$.

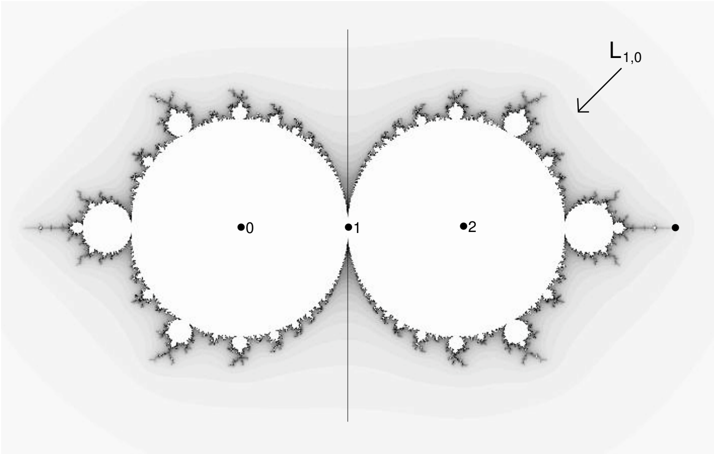

Figure 2: The connectedness locus of $P_{1,\lambda}(z) = \lambda z(1+z)$ and the 0–limb $L_{1,0}$. This is the well known *logistic family* (see, for example, [De]) and its connectedness locus is a double covering of the Mandelbrot set.

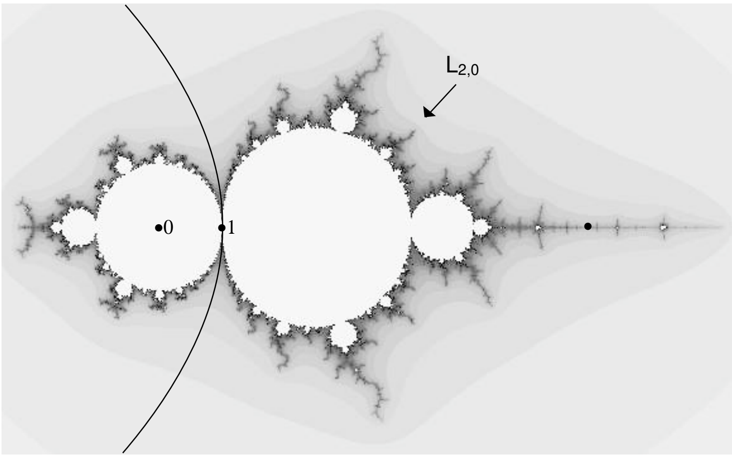

Figure 3: The connectedness locus of $P_{2,\lambda}$ and the 0–limb $L_{2,0}$.

We call a connected component, Ω, of the interior of M or L_q a *hyperbolic component* if for all parameters in Ω, the corresponding polynomials have an attracting periodic orbit. These

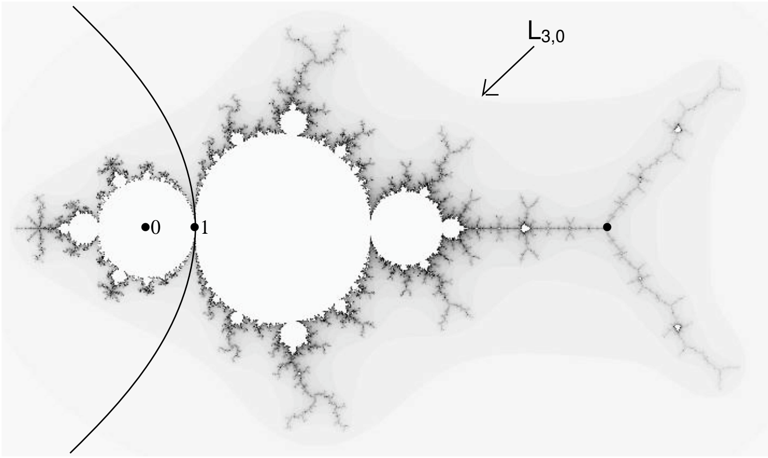

Figure 4: The connectedness locus of $P_{3,\lambda}$ and the 0–limb $L_{3,0}$.

polynomials are hyperbolic in the case of M and subhyperbolic in the case of L_q (except for the component which consists of the unit disc). The period of the attracting orbit must be the same for all parameters in Ω. Hence, we may call it the *period of* Ω.

For each hyperbolic component Ω, there exists a conformal isomorphism $\rho_\Omega : \mathbb{D} \to \Omega$ such that Q_c (resp. $P_{q,\lambda}$) have an attracting cycle of multiplier t if and only if c (resp. λ) equals $\rho_\Omega(t)$ (see [DH2]). This isomorphism extends to the boundary and we thus obtain a parametrization of $\partial\Omega$ by $\gamma_\Omega(t) = \lim_{r\to 1} \rho_\Omega(re^{2\pi i t})$. For each point $t \in \mathbb{R}/\mathbb{Z}$, the point $\gamma_\Omega(t) \in \partial\Omega$ is said to have *internal argument* t.

The largest hyperbolic component in M, denoted by Ω_0, is the component bounded by the main cardioid, and it corresponds to those polynomials that have an attracting fixed point. Attached to each point $c = \gamma_\Omega(p/q) \in \partial\Omega_0$, where $0 < p/q < 1$ is in lowest terms, there is a unique hyperbolic component of period q which we denote by $\Omega_{p/q}$ (see Figure 1). We define the *p/q–limb of M* as the connected component of $M \setminus \overline{\Omega_0}$ attached to the point $\gamma_{\Omega_0}(p/q)$, union this point, and we denote it by $M_{p/q}$. It has been shown that the Mandelbrot set is the union of $\overline{\Omega_0}$ with all the limbs of M (see [La] and [Y]).

The unit disc plays the same role in L_q as Ω_0 in M. Indeed, if $|\lambda| < 1$, the point $z = 0$ is an attracting fixed point of multiplier exactly λ. Hence, in this case, $\rho_{\Omega_0} \equiv \mathrm{Id}$ and $\gamma_\Omega(t) = e^{2\pi i t}$. The difference with Ω_0 resides in the fact that, also at the point $\lambda = 1 = \gamma_{\mathbb{D}}(0)$, there is attached a hyperbolic component, naturally of period one. Analogously to the case of the Mandelbrot set, we define the *r/s–limb of L_q* as the connected component of $L_q \setminus \overline{\mathbb{D}}$ attached to the point $\lambda = \gamma_{\mathbb{D}}(r/s)$, union this point, and we denote it by $L_{q,r/s}$. Of special importance in this paper will be the 0–limb $L_{q,0}$.

The main theorem reads as follows.

Theorem A *Let p and q be positive integers such that $p < q$, $q \geq 2$ and* $\gcd(p,q) = 1$.

Then, there exists a homeomorphism

$$\phi_{p/q} : M_{p/q} \longrightarrow L_{q,0},$$

which is holomorphic in the interior of $M_{p/q}$.

By composing the maps in Theorem A, one obtains several interesting corollaries that relate the limbs of the Mandelbrot set either to themselves or to others with the same denominator (see Figure 5).

Corollary A.1 *Let* p/q *and* p'/q *be as in Theorem A. There exists a homeomorphism*

$$\Phi = \Phi^q_{pp'} : M_{p/q} \longrightarrow M_{p'/q}$$

where $\Phi^q_{pp'} := \phi^{-1}_{p'/q} \circ \phi_{p/q}$. *This map is holomorphic in the interior of* $M_{p/q}$.

Several years ago J-C. Yoccoz observed (unpublished) that the 1/5–limb and the 2/5–limb of the Mandelbrot set were homeomorphic, and that, probably, corresponding polynomials would have conjugate dynamics on the filled Julia set. As we will see, the homeomorphisms $\Phi^q_{pp'}$ do not relate polynomials with conjugate dynamics but they instead have another remarkable property.

Definition Let K_1 and K_2 be two compact sets of $\mathbb{C}$ and $h : K_1 \to K_2$ a homeomorphism. We say that h is *compatible (resp. reversely compatible)* with the embeddings of K_1 and K_2 in the plane if there exist U_1 and U_2 neighborhoods of K_1 and K_2 such that h extends to a homeomorphism $h' : U_1 \to U_2$, preserving (resp. reversing) orientation.

We note that a homeomorphism between two limbs of M that preserved the dynamics (homeomorphisms with this property have been announced in [LS]) could not possibly be compatible with the embedding of the limbs in $\mathbb{C}$. Indeed, they would have to change the cyclic order of the antenae at every branching point. The homeomorphisms we construct do not relate polynomials with conjugate dynamics (although their dynamics are related) but they are compatible with the embeddings of the limbs in the plane (see Sect. 4).

The 0–limbs of the connectedness loci L_q are obviously symmetric with respect to the real axis. If we combine this property with Theorem A we obtain the following corollary.

Corollary A.2 *Let* p, q *be as in Theorem A, and* $C(z) = \overline{z}$. *Then, the homeomorphism*

$$\mathcal{I}_{p/q} := \phi^{-1}_{p/q} \circ C \circ \phi_{p/q} : M_{p/q} \longrightarrow M_{p/q},$$

is an involution (i.e., $\mathcal{I}^2_{p/q} = Id$*) and antiholomorphic in the interior of* $M_{p/q}$. *The fixed points of* $\mathcal{I}_{p/q}$ *form a topological arc through the limb.*

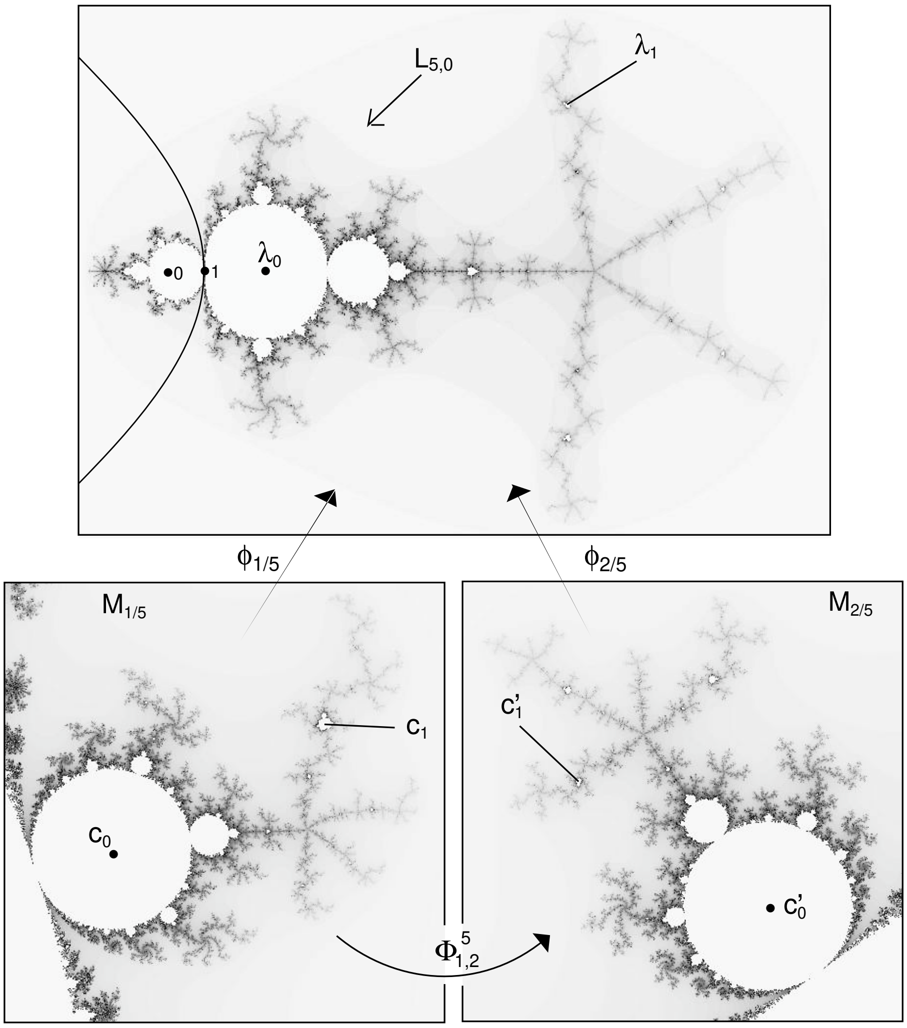

Figure 5: The composition of the maps in Theorem A to obtain the homeomorphism between the 1/5–limb and the 2/5–limb of M. Corresponding points are c_0, λ_0 and c_0', centers of hyperbolic components of periods 5, 1 and 5 respectively. Also c_1, λ_1 and c_1', centers of hyperbolic components of periods 6, 2 and 8 respectively. See Figures 10 and 11 for their filled Julia sets.

One can see [BF, Sect. 6] that these involutions can also be obtained by combining the homeomorphisms of Theorem A with the symmetry of M with respect to complex conjugation. More precisely,

$$\mathcal{I}_{p/q} = \Phi^q_{(q-p)p} \circ C = C \circ \Phi^q_{p(q-p)}.$$

The fixed points are the image under $\phi_{p/q}$ of the points in $L_{q,0}$ that belong to the real axis, and hence a topological arc bissecting the limb. Similar arcs have been shown to exist in [BD], supporting the conjecture of the arcwise connectivity of the Mandelbrot set. Although in principle it could be possible to obtain the homeomorphisms of Corollary A.1 (and hence the involutions) from direct surgery on quadratic polynomials, it is not clear that, without reference to the polynomials $P_{q,\lambda}$, the existence of these arcs would follow.

From the proof of Theorem A one obtains several results on the dynamical planes. They relate the filled Julia sets of parameters in $M_{p/q}$ with their corresponding ones in $M_{p'/q}$ under the map $\Phi^q_{pp'}$. Their statements can be found in Section 3.

The paper is structured as follows. Section 2 contains a discussion of the proof of Theorem A, paying special attention to the topological part of the surgery. As it was mentioned, in Section 3 we state some of the results on the dynamical plane. Section 4 deals with the compatibility of $\Phi^q_{pp'}$ with the embeddings of the limbs in $\mathbb{C}$ (work in progress), and includes some combinatorial results on external rays of M. Finally, Section 5 mentions some conjectures and open questions related to this topic.

2 Proof of Theorem A

In this section we will discuss the proof of Theorem A. As usual, the hard work needs to be done in the dynamical plane. Consider $c \in M_{p/q}$ and, from now on, assume that c is not equal to the root point $\gamma_{\Omega_0}(p/q)$. We must modify the space and the map Q_c in order to obtain a polynomial $P_{q,\lambda}$ and, in particular, a value of $\lambda \in L_q$. This is done by a surgery procedure. This process, after mapping the root point of $M_{p/q}$ to the root point of $L_{q,0}$, defines the map $\phi_{p/q}$. Finally one needs to show the continuity and bijectivity of $\phi_{p/q}$.

2.1 Surgery

The steps of the surgery procedure are as follows.

1. **Topological surgery** We start in the dynamical plane of Q_c for a fixed $c \in M_{p/q}$. Through cutting and sewing (i.e., making some identifications) we construct a new "truncated" space, $\mathbb{C}^T$, and a new map $f_c^{(1)}$, which is the *first return map*. This map has several lines of discontinuity but already exhibits the combinatorial and topological properties of a polynomial $P_{q,\lambda}$.

2. **Quasiconformal surgery** We modify $f_c^{(1)}$ in neighborhoods of the lines of discontinuity (called *sectors*) and obtain a $\mathcal{C}^1$ map, $f_c^{(2)}$, which is quasiregular.

3. Holomorphic smoothing We use the Measurable Riemann Mapping Theorem to obtain a holomorphic map $f_c^{(3)}$ which is polynomial–like of degree $q+1$. Finally, the Straightening Theorem gives the required polynomial $P_{q,\lambda}$.

For simplicity we explain each step for the particular case of $p/q = 1/3$, which already contains all the important points of the proof of the general case [BF, Sect. 5.2].

Dynamical characterizations of $\mathbf{Q}_c$ with $\mathbf{c} \in M_{1/3}$

Let $K_c = K(Q_c)$ and $J_c = J(Q_c)$. Since $c \notin \overline{\Omega}_0$, the two fixed points of Q_c are repelling. Let β_c denote the most repelling one, which is always the landing point of the 0–ray. Let α_c denote the remaining fixed point.

Since $c \in M_{1/3}$, there are three external rays landing at α_c. The arguments of these rays must form a period three cycle under the doubling map, and have rotation number $1/3$. These conditions determine that the external rays must be $R_c(1/7)$, $R_c(2/7)$ and $R_c(4/7)$. Because of the symmetry with respect to the origin, $-\alpha_c$ is the landing point of the remaining preimages of these rays which are $R_c(1/14)$, $R_c(9/14)$ and $R_c(11/14)$.

The rays landing at α_c and $-\alpha_c$ divide the plane into five closed subsets (see Figure 6). We denote by V_c^0 the one that contains the critical point $\omega = 0$, and the others V_c^1, V_c^2 and $\widetilde{V}_c^1 = -V_c^1$ and $\widetilde{V}_c^2 = -V_c^2$ as shown in Figure 6. Each of these subsets contains a connected component of $K_c \setminus \{\alpha_c, -\alpha_c\}$ in its interior.

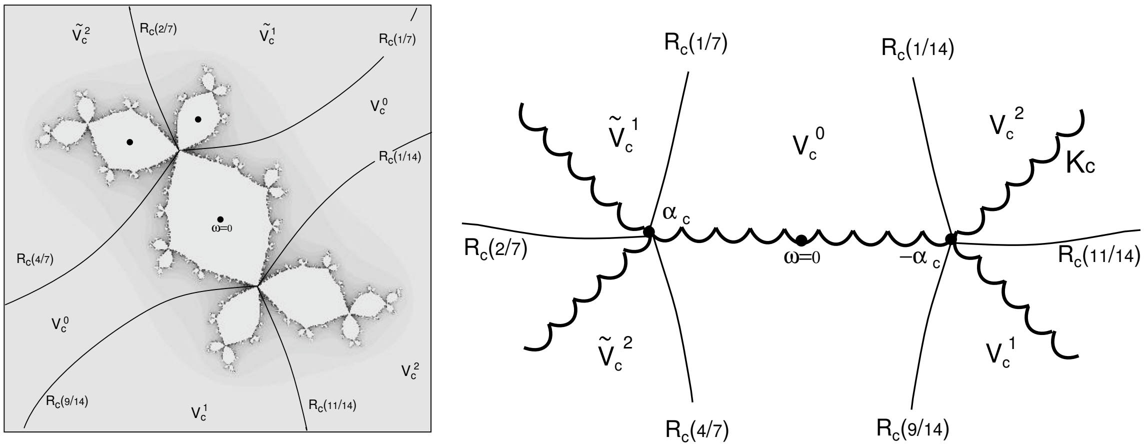

Figure 6: Filled Julia set of Q_c for $c \in M_{1/3}$ and sketch of a typical one.

Looking at the dynamics of the rays, we see that Q_c acts on this partition as follows:

$$\begin{array}{lcl} V_c^0 & \xrightarrow{2-1} & \widetilde{V}_c^1 \\ V_c^1, \widetilde{V}_c^1 & \xrightarrow{1-1} & \widetilde{V}_c^2 \\ V_c^2, \widetilde{V}_c^2 & \xrightarrow{1-1} & V_c^0 \cup V_c^1 \cup V_c^2 \end{array}$$

Dynamical characterizations of $\mathbf{P}_{\mathbf{3},\lambda}$ for $\lambda \in L_{3,0}$

Polynomials in $L_{3,0}$ share the property of having the fixed point $z = 0$ as the landing point of only one fixed ray, chosen to be $R_\lambda(0)$ (see [BF, Sect. 3.1] for details on the fact that this is not a monic family of polynomials). Since the only preimage of 0 apart from itself is the critical point -3, all other preimages of $R_\lambda(0)$, namely

$$R_\lambda(1/4), R_\lambda(1/2), R_\lambda(3/4),$$

must land at -3.

These three rays divide the complex plane into three close subsets which we denote by V_λ^0, V_λ^1 and V_λ^2 as shown in Figure 7. There is a connected component of $K_\lambda \setminus \{-3\}$ in each of these sets.

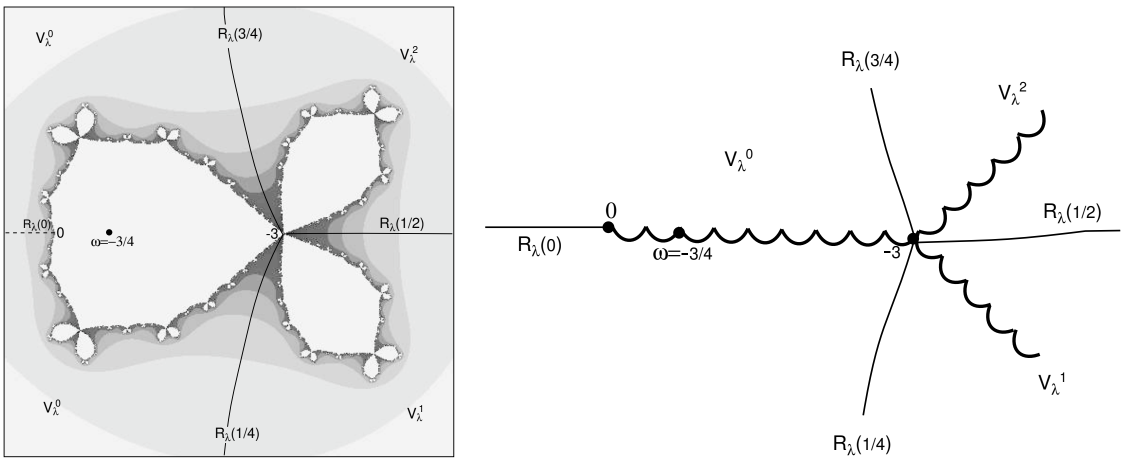

Figure 7: Filled Julia set of $P_{3,\lambda}$ for $c \in L_{3,0}$ and sketch of a typical one. These dynamical planes have been rotated 180 degrees.

The other critical point $\omega = -3/4$ lies in the interior of V_λ^0. Hence, $P_{3,\lambda}$ acts on these sets as follows.

$$\begin{array}{rcl} \overset{\circ}{V}{}_\lambda^0 & \xrightarrow{2-1} & \mathbb{C} \setminus (R_\lambda(0) \cup \{0\}) \\ \overset{\circ}{V}{}_\lambda^1, \overset{\circ}{V}{}_\lambda^2 & \xrightarrow{1-1} & \mathbb{C} \setminus (R_\lambda(0) \cup \{0\}) \end{array}$$

Topological surgery

Given $c \in M_{1/3}$, we want to modify the dynamical plane of Q_c in order to create a new map with the dynamical characterizations of a polynomial $P_{3,\lambda}$ for some $\lambda \in L_{3,0}$. In order to do that, we cut along the rays $R_c(1/7)$ and $R_c(4/7)$ and identify them equipotentially (see Figure 8). That is, we define the *truncated* complex plane as

$$\mathbb{C}_c^T = (V_c^0 \cup V_c^1 \cup V_c^2)/\sim$$

where $\sim$ is the equivalence relation that identifies a point $z \in R_c(1/7)$ with the unique point $z' \in R_c(4/7)$ such that $G_c(z) = G_c(z')$. This new space is a Riemann surface isomorphic to $\mathbb{C}$. Note that no identification takes place in the filled Julia set. We define the *truncated filled Julia set* as

$$K_c^T = K_c \cap \mathbb{C}_c^T = K_c \cap (V_c^0 \cup V_c^1 \cup V_c^2).$$

We proceed to define a new map $f_c^{(1)}$ on $\mathbb{C}_c^T$ which is the first return map on this space, that is

$$f_c^{(1)}(z) = \begin{cases} Q_c^3(z) & \text{if } z \in V_c^0, \\ Q_c^2(z) & \text{if } z \in V_c^1, \\ Q_c(z) & \text{if } z \in V_c^2. \end{cases}$$

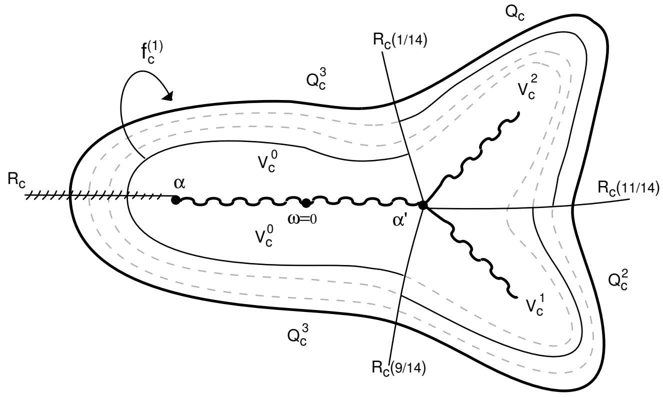

Figure 8: The truncated complex plane $\mathbb{C}_c^T$ and the first return map. An equipotential curve and its preimage under $f_c^{(1)}$ have been drawn to show the shift discontinuity at the rays landing at $-\alpha_c$.

Note that the map is not well defined on the rays $R_c(1/14)$, $R_c(9/14)$ and $R_c(11/14)$, since a shift discontinuity occurs.

The reader should view the sets V_c^i for $i = 0, 1, 2$ as those that will become the sets V_λ^i.

Indeed, the first return map acts on these sets by mapping

$$\begin{aligned} \overset{\circ}{V}{}^0_c &\overset{2-1}{\longrightarrow} \mathbb{C} \setminus (R_c \cup \{\alpha_c\}) \\ \overset{\circ}{V}{}^1_c, \overset{\circ}{V}{}^2_c &\overset{1-1}{\longrightarrow} \mathbb{C} \setminus (R_c \cup \{\alpha_c\}), \end{aligned}$$

where R_c denotes the two identified rays.

Note that $f_c^{(1)}$ is well defined on R_c. It is also holomorphic wherever defined, except at $-\alpha_c$. However, $f_c^{(1)}$ already has many of the topological and combinatorial properties of a polynomial $P_{3,\lambda}$ for some $\lambda \in L_{3,0}$. We have just seen some of them but, moreover,

- The map $f_c^{(1)}$ has topological degree four, i.e., every point in $\mathbb{C}_c^T \setminus R_c$ has four preimages (counting multiplicity).
- There is a new critical point at $-\alpha_c$ with multiplicity two. Intuitively, this can be checked by seeing how the image of any small neighborhood of $-\alpha_c$ wraps three times around the critical value α_c. This critical point is prefixed.

Quasiconformal surgery

We will outline the process where we modify $f_c^{(1)}$ to obtain a quasiregular map $f_c^{(2)}$, which still preserves the properties of $f_c^{(1)}$. Recall that a *quasiregular* map is a quasiconformal map which is allowed to have critical points. More precisely, it is a composition of a quasiconformal homeomorphism with a holomorphic map.

We start by modifying $f_c^{(1)}$ on a neighborhood of the rays of discontinuity. These neighborhoods are contained in $\mathbb{C}_c^T \setminus K_c^T$ and called *sectors* (see Figure 9). They are actually defined on the complement of the unit disc (or equivalently, on the right half plane) and then pulled back by the Böttcher coordinate ψ_c (see [BF, Sect. 4.1.1] for details). We denote these sectors by $S_c(1/14)$, $S_c(9/14)$ and $S_c(11/14)$ respectively, and let S'_c be equal to their union. All points in $S'_c c$ are mapped under $f_c^{(1)}$ to a sector around R_c (by choice) which we denote by S_c.

In order for these sectors not to overlap, we must restrict to a bounded set around K_c^T, namely the region enclosed by an equipotential curve γ_c of fixed potential $\eta > 0$. Let X_c be this region. We define a simple curve γ'_c in X_c which, outside the sectors, coincides with the preimage of γ_c under $f_c^{(1)}$, i.e., some disjoint pieces of equipotential curves. The segments inside S'_c, are chosen to be $\mathcal{C}^\infty$ curves that join the two disconnected ends (see Figure 9). Let X'_c be the domain bounded by γ'_c.

The map $f_c^{(2)} : X'_c \to X_c$ is defined as follows.

$$f_c^{(2)}(z) = \begin{cases} f_c^{(1)}(z) & \text{if } z \notin S'_c \\ g(z) & \text{if } z \in S'_c \end{cases}$$

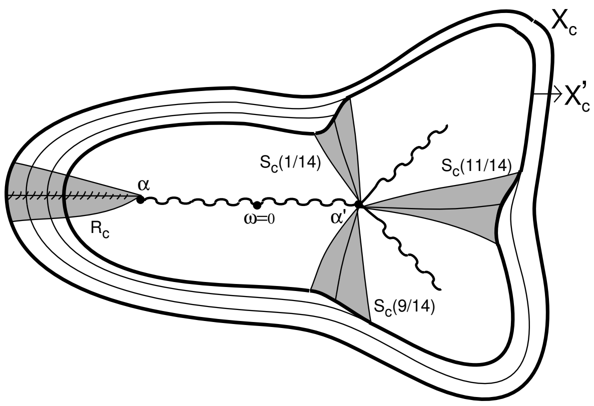

Figure 9: The sectors defined around the rays of discontinuity, and the domain and image of $f_c^{(2)}$.

where g is chosen to be a $\mathcal{C}^1$ diffeomorphism such that it defines the same tangent map as $f_c^{(1)}$ on the boundary of the sectors. In this definition, a pullback process is used (see [BF, Sect. 5.2] for details), so that $f_c^{(2)}$ is quasiregular or, equivalently, the field of ellipses $E_x = (T_x f_c^{(2)})^{-1}(S^1)$ has bounded dilatation ratio.

Remark 2.1 An essential point for this construction to work is the fact that orbits enter the sectors at most once (Shishikura principle on surgery). Since the map is holomorphic everywhere else, the bound on the dilatation ratio is constant everywhere.

Holomorphic smoothing and definition of $\phi_{\mathbf{p/q}}$

The next step is to obtain a holomorphic map $f_c^{(3)}$ with the same properties as $f_c^{(2)}$ but which is polynomial–like map of degree four. We then can apply the Straightening Theorem (see [DH3] or [BF, Sect. 2.2]) to obtain an actual polynomial $P_{3,\lambda}$.

The procedure goes as follows.

- Construct a measurable field of ellipses (or an *almost complex structure*) σ_c on X_c with bounded dilatation ratio which is invariant under $f_c^{(2)}$, and which consists of circles on K_c^T.

- Apply the Measurable Riemann Mapping Theorem (see [A] or [BF, Sect. 2.2]) to obtain a quasiconformal homeomorphism $\varphi_c : \overset{\circ}{X}_c \to \mathbb{D}$ that integrates σ_c, i.e., that makes the map
$$f_c^{(3)} = \varphi_c \circ f_c^{(2)} \circ \varphi_c^{-1} : D'_c \longrightarrow \mathbb{D}$$
holomorphic on $D'_c = \varphi_c(\overset{\circ}{X'_c})$.

- Since φ_c is a homeomorphism, $f_c^{(3)}$ is still a ramified covering of degree four with two critical points. Moreover D'_c is relatively compact in $\mathbb{D}$, hence the map $f_c^{(3)}$ is polynomial-like of degree four. We may hence apply the Straightening Theorem to obtain a degree four actual polynomial P and a hybrid equivalence χ_c that conjugates $f_c^{(3)}$ to
$$P = \chi_c \circ f_c^{(3)} \circ \chi_c^{-1}$$
on neighborhoods of K_c^T and $K(P)$ respectively.

- Finally, we check that P has all the properties of a polynomial in the family $P_{3,\lambda}$, with $\lambda \in L_{3,0}$. Hence, after conjugating by an affine map we may assume that the polynomial obtained is of this form, for some $\lambda = \lambda(c)$. One can see that the value of λ does not depend on the choices made during the construction. Thus, we can define
$$\phi_{1/3} : M_{1/3} \longrightarrow L_{3,0}$$
by setting $\phi_{1/3}(c) = \lambda(c)$, and mapping the root point to the root point.

2.2 The map $\phi_{\mathbf{p/q}}$ is a homeomorphism

In this section we comment on the necessary steps to prove that the map defined above is actually a homeomorphism which is moreover analytic in the interior of the limbs.

Bijectivity

The bijectivity of $\phi_{p/q}$ is proven by explicitely constructing an inverse map
$$\xi_{p/q} : L_{q,0} \longrightarrow M_{p/q},$$
and showing that the composition both ways equals the identity. The construction of the inverse is done again by a surgery procedure which, intuitively, consists on "undoing" everything that was done before. Hence this time we must "add" to the dynamical plane of a polynomial $P_{q,\lambda}$, with $\lambda \in L_{q,0}$, the parts of the plane that were removed in the construction of $\phi_{p/q}$, and define a new map accordingly. Once the analogous to the first return map has been defined, the remaining steps of surgery are as above.

To show that the composition of both maps is equal to the identity, it is important to note that, as it is the case of quadratic polynomials Q_c, the family of polynomials $P_{q,\lambda}$ contains a single element of each hybrid equivalence class, that is, if $P_{q,\lambda}$ and $P_{q,\lambda'}$ are hybrid equivalent then $\lambda = \lambda'$ [BF, Prop. 3.3]. Hence the proof consists of showing that the polynomials $P_{q,\lambda}$ and $P_{q,\lambda'}$ where $\lambda' = \phi_{p/q} \circ \xi_{p/q}(\lambda)$ are hybrid equivalent and that the same holds for Q_c and $Q_{c'}$ where $c' = \xi_{p/q} \circ \phi_{p/q}(c)$. This is accomplished by following the surgery procedures and making the right choices for the boundaries and sectors at each step (see [BF, Sect. 5.3] for details).

Analyticity in the interior of the limbs

If Ω_M is a hyperbolic component of the interior of M, it is easy to see that $\phi_{p/q}$ is analytic in Ω_M. Indeed, by following the surgery procedure, we see that if $\omega = 0$ is attracted to a periodic cycle by Q_c then the critical point $\omega_q = -q/(q+1)$ is attracted to a periodic cycle by $P_{q,\lambda}$ where $\lambda = \phi_{p/q}(c)$. The periods of the cycles are not equal but their multipliers coincide. Hence, $\Omega_L = \phi_{p/q}(\Omega_M)$ is a hyperbolic component of $L_{q,0}$ and moreover

$$\phi_{p/q}|_{\Omega_M} = \rho_{\Omega_L} \circ \rho_{\Omega_M}^{-1},$$

where $\rho_{\Omega_L} : \mathbb{D} \to \Omega_L$ and $\rho_{\Omega_M} : \mathbb{D} \to \Omega_M$ are the respective parametrizations given by the multiplier maps. Since these maps are conformal isomorphisms, this shows that $\phi_{p/q}$ is holomorphic on Ω_M. A paralel statement holds for $\xi_{p/q}$.

It is not yet known that all components of the interior of M are hyperbolic (this is known as the hyperbolicity conjecture and it would be a consequence of MLC). Some of them might be *non–hyperbolic* or (as they are frequently called) *queer components.* If c belongs to a non–hyperbolic component, it is known that K_c must have empty interior and J_c must be of positive measure.

To show that $\phi_{p/q}$ is holomorphic in a non–hyperbolic component we use the same argument as above but with a different parametrization, namely the deformations of the unique (up to rotation) invariant line field of Q_c and $P_{q,\lambda}$ (see [BF, Sect. 5.4] for details).

Continuity at points on the boundary

Given a sequence $c_n \to c \in \partial M_{p/q}$, let $\lambda_n = \phi_{p/q}(c_n)$ and $\lambda = \phi_{p/q}(c)$. To prove continuity at c we need to show that $\lambda_n \to \lambda$ or, equivalently (since $L_{q,0}$ is compact), that any convergence subsequence $\lambda_{n_k} \to \widetilde{\lambda}$ satisfies $\widetilde{\lambda} = \lambda$.

A key property of the family of polynomials $P_{q,\lambda}$ is the following.

Lemma 2.2 (BF, Lemma 5.22) *Suppose $\lambda \in \partial L_q$, $\lambda' \in \mathbb{C}$ and the polynomials $P_{q,\lambda}$ and $P_{q,\lambda'}$ are quasiconformally conjugate. Then, $\lambda' = \lambda$.*

This is known to hold for the family Q_c (see [DH3]) and it makes it sufficient to show that the polynomials $P_{q,\lambda}$ and $P_{q,\widetilde{\lambda}}$ are quasiconformally conjugate (see [BF, Lemma 5.25]) in order to conclude continuity.

We remark that this argument works only at points of the boundary, since the lemma is not true for points in the interior of M, and no more than quasiconformal conjugacy can be shown. For interior points, we are not able to show continuity directly, i.e., without going through the multiplier map.

An important point in the proof of continuity is to realize that most of the choices in the surgery construction (the boundary of the truncated space, the almost–complex structure, etc.) have been done *once and for all* on the complement of the unit disc (or in fact, on the right half plane) and then pulled back by (a restriction of) the Böttcher coordinates ψ_c. Since the map ψ_c varies analytically with respect to c, it follows that the truncated space, the sectors and the map $f_c^{(2)}$ vary analytically with c. This is not a precise statement and it needs to be formalized by the use of holomorphic motions and the λ–lemma (see [BF, Lemma 5.24]).

3 Results on the dynamical plane

From the proof of Theorem A we obtain the following results on the dynamical plane.

Theorem B (BF, Thm. G) *Let $c \in M_{p/q}$ and $\lambda = \phi_{p/q}(c) \in L_{q,0}$. Let W_λ be a neighborhood of $K_\lambda = K(P_{q,\lambda})$ which is the filled level set of a chosen equipotential curve. There exists a homeomorphism*

$$H_c : X_c \longrightarrow W_\lambda,$$

which satisfies $\overline{\partial} H_c = 0$ on K_c^T and conjugates $f_c^{(2)}$ to $P_{q,\lambda}$.

The homeomorphism H_c is basically the composition of the integrating map φ_c and the straightening map χ_c. However, in this composition one must add a hybrid equivalence that makes sure that relevant sectors, boundaries, etc. are sent to their corresponding ones. The only reason for this is to deduce the following corollary.

Corollary B.1 *Let $\Phi^q_{pp'} : M_{p/q} \to M_{p'/q}$ be as in Corollary A.1, and let $c \in M_{p/q}$ and $c' = \Phi^q_{pp'}(c)$. There is a homeomorphism*

$$\widehat{\Phi}_c : K_c \longrightarrow K_{c'}$$

which is holomorphic in the interior of K_c and compatible with the embeddings of K_c and $K_{c'}$ in $\mathbb{C}$.

(Other similar results can be obtained from the involutions in Corollary A.2). To construct this map one first defines $\widehat{\Phi}_c^T : X_c \setminus S_c \to X_{c'} \setminus S_{c'}$ as $\widehat{\Phi}_c^T = H_c \circ H_{c'}^{-1}$ where S_c and $S_{c'}$ are the sectors around the identified rays and H_c, $H_{c'}$ are given by Theorem B. Afterwards, set

$$\widehat{\Phi}_c = \begin{cases} \widehat{\Phi}_c^T(z) & \text{if defined} \\ -\widehat{\Phi}_c^T(-z) & \text{otherwise.} \end{cases}$$

We refer to [BF, Sect. 5.6] for details and to Figures 10 and 11 for some examples of corresponding Julia sets.

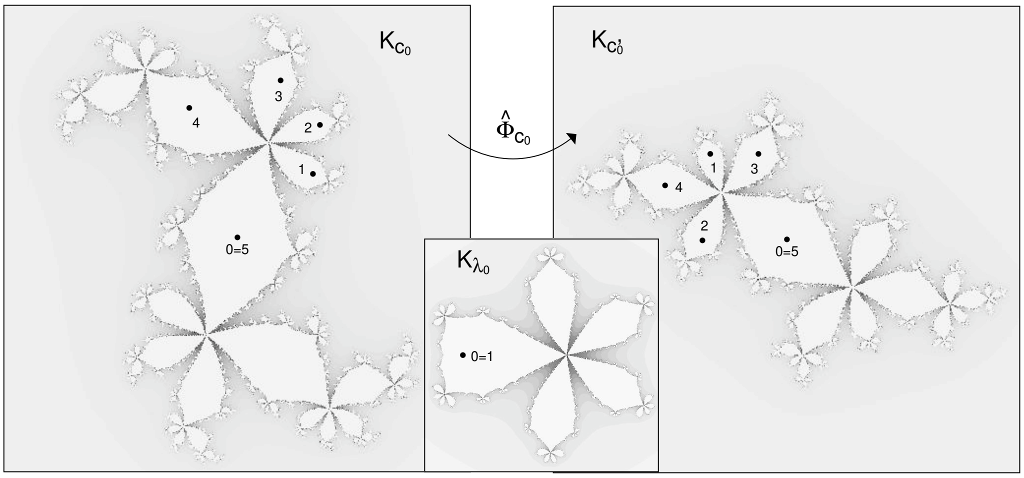

Figure 10: From left to right: filled Julia sets of Q_{c_0}, P_{5,λ_0} and $Q_{c'_0}$ where $c_0 \overset{\phi_{1/5}}{\longrightarrow} \lambda_0 \overset{\phi_{2/5}^{-1}}{\longrightarrow} c'_0$. (Compare to Figure 5).

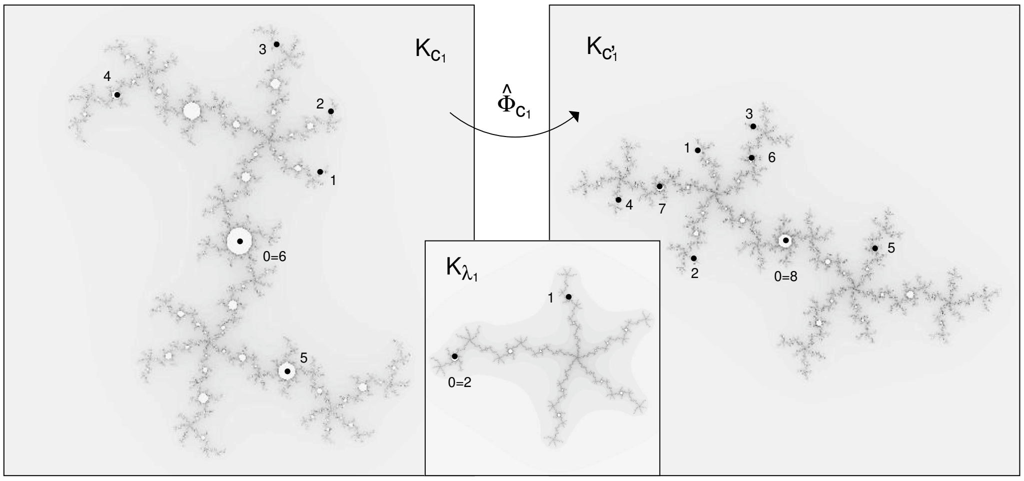

Figure 11: From left to right: filled Julia sets of Q_{c_1}, P_{5,λ_1} and $Q_{c'_1}$ where $c_1 \overset{\phi_{1/5}}{\longrightarrow} \lambda_1 \overset{\phi_{2/5}^{-1}}{\longrightarrow} c'_1$. (Compare to Figure 5).

4 Compatibility with the embeddings

To prove that the homeomorphisms $\Phi^q_{pp'} : M_{p/q} \to M_{p'/q}$ are compatible with the embeddings of $M_{p/q}$ and $M_{p'/q}$ in $\mathbb{C}$, it is necessary to extend the map $\Phi^q_{pp'}$ to a neighborhood of the limbs. The first temptation is to define this extension in the most intuitive way, namely mapping external rays to external rays equipotentially, in such a way that it "matches" with $\Phi^q_{pp'}$ on the boundary of the limbs. In fact, we only require a map on external arguments.

Theorem 4.1 (BF, Thm. C) *Let $\theta^{\pm}_{p/q}$ and $\theta^{\pm}_{p'/q}$ be the arguments of the external rays landing at the root points of the limbs $M_{p/q}$ and $M_{p'/q}$, respectively. Then there exists an orientation preserving homeomorphism*

$$\Theta^q_{pp'} : [\theta^-_{p/q}, \theta^+_{p/q}] \longrightarrow [\theta^-_{p'/q}, \theta^+_{p'/q}]$$

such that for $\theta \in [\theta^-_{p/q}, \theta^+_{p/q}] \cap \mathbb{Q}$, the ray $R_M(\theta)$ lands at the point $c \in M_{p/q}$ if and only if $R_M(\Theta^q_{pp'}(\theta))$ lands at $\Phi^q_{pp'}(c) \in M_{p'/q}$.

In fact, by construction, the map $\Theta^q_{pp'}$ only has this property for Misiurewicz points. The remaining rational arguments (and many irrational ones) are obtained after taking limits. If M is locally connected then this result extends to all irrational arguments and we obtain:

Theorem 4.2 (BF, Thm. E) *Assume M is locally connected. Then, the map $\Phi^q_{pp'}$ is compatible with the embeddings of the limbs in $\mathbb{C}$.*

However, the compatibility with the embeddings is true without the assumption of local connectivity, as we will show in a forthcoming paper. The extension of $\Phi^q_{pp'}$ to a neighborhood of the limbs must be done in a different way.

To define the map $\Phi^q_{pp'}$ for values of c outside of $M_{p/q}$ (but in a small neighborhood), we again need to perform surgery as we did for the proof of Theorem A. Even if the Julia sets are now disconnected, in principle there is no problem in repeating the process except for two facts.

The first one is the definition of the sectors. When K_c is connected, the sectors are defined in the right half plane and then pulled back by the Böttcher coordinates so that the continuity with respect to c is assured. But when c is outside of the Mandelbrot set, the Böttcher coordinates are no longer continuously defined in the whole complement of K_c, but only for points with potential larger than $G_c(c)$. To solve this problem, we have to define the sectors on the right half plane as before but only up to an equipotential curve of fixed potential $\eta > G_c(c)$. We can then pull back by the Böttcher coordinate and define the remaining pieces of the sectors in a dynamical way *directly* on the dynamical plane of Q_c. In this way, we assure that the sectors will not intersect the Julia set.

We find the second obstacle once we have defined the required polynomial like map $f_c^{(3)}$ and we want to obtain the actual polynomial $P_{q,\lambda}$. The Straightening Theorem also applies to polynomial–like maps with a disconnected Julia set but, in that case, the resulting polynomial is no longer unique. As a result, although one can always make a choice that assures the continuity of the Straightening map, the resulting extension of $\Phi^q_{pp'}$ will not necessarily be injective. Fortunately, this problem has motivated a recent result of X. Buff who, by means of the Douady–Earle extension, has shown that a continuous extension (plus some conditions as the ones given in Theorem 4.1) is sufficient in order to show the compatibility of the embeddings. (See [Bu]).

5 Conjectures and open problems

In this section we state some questions and conjectures related to the topics in this paper.

Quasiconformality of the homeomorphisms

Conjecture 1 *The homeomorphisms $\Phi^q_{pp'}$ are quasiconformal.*

Recall that the maps $\Phi^q_{pp'}$ are holomorphic in the interior of the limbs but that only continuity has been proven at points on the boundary of the limbs. This conjecture is based on the proof of M. Lyubich that primitive copies of M (those with a cusp) are quasiconformal copies (see [Ly]).

Other limbs of $\mathbf{L}_q$

Throughout the paper we have only considered the 0–limb of each connectedness locus L_q. Let $L_{q,r/s}$ denote the r/s–limb of L_q as defined in Section 1. For a polynomial $P_{q,\lambda}$ with $\lambda \in L_{q,r/s}$, the point $z = 0$ is still a repelling fixed point but this time is the landing point of a cycle of s external rays. Indeed, $z = 0$ experiments an s–tupling bifurcation when λ is at the point in the unit circle with internal argument r/s. Consequently, the critical point $-q$ is now the landing point of $q \cdot s$ external rays, exactly the remaining preimages of the rays landing at 0.

Conjecture 2 *Let r and s be positive integers such that $r < s$ and $\gcd(r, s) = 1$. For any positive integers p and q such that $p < q\cdot s$ and $\gcd(p, q\cdot s) = 1$, there exists a homeomorphism*

$$h = h(r, s, p, q) : M_{\frac{p}{q\cdot s}} \longrightarrow L_{q,\frac{r}{s}}.$$

The combinatorial or topological part of the surgery procedure seems to be clear, although we not only need to "cut away" some part of the filled Julia set of K_c but also "add" extra pieces which are copies of the existing ones. This seems to suggest that surgery techniques as those in the second part of [BD] (opening modulus) will need to be used.

Secondary limbs of M

Let Ω be a hyperbolic component of M of period $P > 1$ and let $\gamma_\Omega : S^1 \to \partial\Omega$ be the parametrization of its boundary that sets $\gamma_\Omega(0)$ equal to the root of Ω. For any $0 < p/q < 1$ in lowest terms we define the $\{\Omega, p/q\}$*–limb*, $\Omega_{p/q}$ as the connected component of $M \setminus \overline{\Omega}$ attached to $\partial\Omega$ at the point $\gamma_\Omega(p/q)$.

We know that, using a tuning procedure (see [DH3]), there is a homeomorphic copy of the Mandelbrot set that corresponds the hyperbolic component Ω with the main cardioid of M. Consequently, each $\{\Omega, p/q\}$–limb contains a copy of the actual p/q–limb of M. This suggests the following question.

Question *Can we apply surgery techniques as above to conclude that all the* $\{\Omega, p/q\}$*–limbs with the same denominator are homeomorphic? Are they related to the limbs of* $L_{q'}$ *for some* q'*?*

Generalizing surgery

It is clear that the technique of surgery on complex polynomials in order to construct homeomorphisms between parameter spaces varies from one case to another only on the topological or combinatorial part. The rest of the process consists of a lot of hard work but seems to be relatively similar in all instances. We believe it could be very useful and interesting to generalize the procedure into a result which, after checking certain hypothesis, could be applied to conclude, for example, the continuity of the constructed homeomorphisms or, even more, the existence of the homeomorphisms themselves.

Exponentials

The family of polynomials $P_{q,\lambda}$ was studied in [F] as a way of approximating the family of entire transcendental functions $G_\lambda(z) = \lambda z \exp(z)$ after letting q tend to infinity. It is clear that the polynomials $P_{q,\lambda}$ tend to the maps G_λ uniformly on compact sets but there are also some kinds of dynamical convergence. For the case of the exponential, this phenomenon was studied in [DGH]. Recently, B. Krauskopf and H. Kriete have obtained several interesting results on convergence of hyperbolic components and other aspects, not only for this family but also for others with similar properties (see for example [KK1, KK2]). Although this is a topic with many remaining open questions, we chose to formulate the following (somehow vague) one, given its relation with the Mandelbrot set.

Question *In view of results like Theorem A and the convergence of* $P_{q,\lambda}$ *to* G_λ*, what can we say about the limit of* $M_{1/q}$ *when* $q \to \infty$ *(known as the limit elephant)?*

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