TRANSLUNAR HALO ORBITS IN THE QUASI-BICIRCULAR PROBLEM

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Abstract. The Quasi-bicircular Problem (QBCP) is a four body problem where three masses are revolving in a quasi-bicircular motion (that is, a coherent motion close to bicircular), the fourth mass being small and not influencing the motion of the three primaries. The Earth-Moon-Sun-Spacecraft case is considered and the Hamiltonian which governs the motion of the fourth mass is derived. That is a Hamiltonian with three degrees of freedom depending periodically on time. One application of this model has been the computation of quasi-periodic translunar Halo orbits. These orbits have been obtained using a continuation method starting at the Halo orbits of the RTBP. The Halo orbits of the QBCP are refined to Halo orbits of the Solar System using a Parallel Shooting method and JPL ephemeris.

1. Quasi-bicircular solution of the three body problem

We are going to derive a planar quasi-bicircular solution of the general three body problem (3BP), using the masses of the Moon, Earth and Sun, which are denoted by μ, $1-\mu$ and m_s, respectively. The units of mass, time and length are taken in such a way that the gravitational constant, the Earth-Moon distance and their mean motion are equal to 1.

We consider the Jacobi formulation of the 3BP using the relative position vectors $\vec{r}$ and $\vec{R}$ (see figure 1). The equations of the motion can be written in complex notation, using z and Z instead of $\vec{r}$ and $\vec{R}$ (see [4]). They are

$$\ddot{z} \;=\; -\frac{z}{r^3} + m_s\left(\frac{Z-\mu z}{r_{es}^3} - \frac{Z+(1-\mu)z}{r_{ms}^3}\right)\,, \tag{1}$$

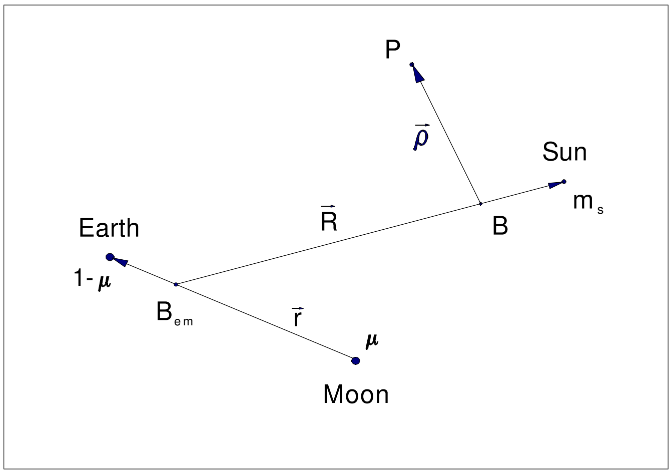

Figure 1. Quasi-bicircular problem in an inertial frame of reference.

$$\ddot{Z} = -(1+m_s)\left(\mu\,\frac{Z+(1-\mu)z}{r_{ms}^3} + (1-\mu)\,\frac{Z-\mu z}{r_{es}^3}\right), \qquad (2)$$

where $r=|z|$, $r_{ms}=|Z+(1-\mu)z|$, $r_{es}=|Z-\mu z|$. We want a solution

$$z \simeq e^{it}\;, \qquad\qquad Z \simeq a_s e^{i n_s t}\;,$$

where n_s is the mean motion of the Sun and a_s the mean distance between Sun and Earth-Moon barycenter. It is possible to find such a solution as

$$z = e^{it}\sum_{j=-\infty}^{+\infty} b_j e^{ij\omega_s t}\;, \qquad\qquad Z = a_s e^{i n_s t}\sum_{j=-\infty}^{+\infty} c_j e^{ij\omega_s t}\;, \qquad (3)$$

being $\omega_s = 1-n_s$ and $b_j, c_j \in \mathbb{R}$. For this purpose, we introduce the small parameter $\epsilon = 1/a_s \simeq 1/389$, and look for a solution of the form

$$z = e^{it}\,(1+\sum_{k\geq 1}\epsilon^k u_k)\;, \qquad\qquad Z = a_s e^{i n_s t}\,(1+\sum_{k\geq 1}\epsilon^k v_k)\;,$$

where u_k and v_k are trigonometric polynomials with basic frequency ω_s. Substituting into the differential equations 1 and 2 and equating terms of the same order in ϵ, the polynomials u_k and v_k can be obtained recursively. To compute each polynomial, a linear differential equation with constant coefficients is solved. Finally, adding all the terms in powers of ϵ, we get the coefficients b_j and c_j of the expressions 3.

2. Quasi-bicircular Problem

It is a four body problem where three masses are revolving in a quasi-bicircular motion, the fourth mass being small and not influencing the motion of the three primaries. In our case, the three primaries are Sun, Earth and Moon, so the vectors $\vec{r}(t)$ and $\vec{R}(t)$ are the functions computed in the previous section. The fourth particle can be, for instance, an spacecraft.

The figure 1 shows the QBCP in an inertial reference system. To express the Hamiltonian of the mass P in a suitable way, we perform a change of coordinates (translation + rotation + homothetic transformation) such that Earth and Moon are located at the fixed points $(\mu, 0, 0)$ and $(\mu - 1, 0, 0)$, respectively, as is usual in RTBP. Then, the Hamiltonian is

$$\begin{aligned} H &= \frac{1}{2}\alpha_1(p_x^2 + p_y^2 + p_z^2) + \alpha_2(p_x x + p_y y + p_z z) + \alpha_3(p_x y - p_y x) + \\ &\quad + \alpha_4 x + \alpha_5 y - \alpha_6 \left(\frac{1-\mu}{q_{pe}} + \frac{\mu}{q_{pm}} + \frac{m_s}{q_{ps}} \right) , \end{aligned}$$

where

$$q_{pe}^2 = (x-\mu)^2 + y^2 + z^2 , \qquad q_{pm}^2 = (x - \mu + 1)^2 + y^2 + z^2 ,$$
$$q_{ps}^2 = (x - \alpha_7)^2 + (y - \alpha_8)^2 + z^2 ,$$

and α_k are of the form

$$\begin{aligned} \alpha_k(t) &= \alpha_{k0} + \sum_{j\geq 1} \alpha_{kj} \cos(j\omega_s t) , \qquad k = 1,3,4,6,7, \\ \alpha_k(t) &= \sum_{j \geq 1} \alpha_{kj} \sin(j\omega_s t) , \qquad k = 2,5,8. \end{aligned}$$

The α_k functions can be computed from $\vec{r}(t)$ and $\vec{R}(t)$ and its derivatives. The table 1 shows some Fourier coefficients of these functions.

Notice that the Hamiltonian of the QBCP can be considered as a perturbation of the RTBP with Earth-Moon mass parameter. The main difference is that the QBCP in non autonomous. It depens periodically on time with frequency ω_s. The period is denoted by $T = 2\pi/\omega_s$.

3. Quasi-periodic Halo Orbits in the QBCP

We want to obtain an orbit of the form

$$x = \sum_{k\in Z^2} x_k \gamma_1^{k_1}\gamma_2^{k_2} , \qquad y = i \sum_{k\in Z^2} y_k \gamma_1^{k_1}\gamma_2^{k_2} , \qquad z = \sum_{k\in Z^2} z_k \gamma_1^{k_1}\gamma_2^{k_2} ,$$

TABLE 1. Greatest Fourier coefficients of α_k functions. See more in [1].

j	α_1 c	α_2 s	α_3 c	α_4 c
0	1.00184161e+00		1.00000000e−00	−9.75524233e−04
1	5.76751773e−04	−2.64437603e−04	5.63412600e−04	2.15476436e+00
2	1.43877703e−02	−1.32868690e−02	1.88968744e−02	3.65748447e−04
3	−2.63036297e−06	9.38609321e−06	−9.91175880e−06	3.29567338e−03
4	1.17627836e−04	−1.21850906e−04	1.56870814e−04	3.30103140e−07
5	−8.06858139e−08	1.52212760e−07	−1.70776258e−07	1.27884069e−05
6	9.84324977e−07	−1.07210266e−06	1.31961368e−06	−2.62379795e−09
7	−1.17205439e−09	1.88937126e−09	−2.13655004e−09	6.53380551e−08
8	8.31190597e−09	−9.32498503e−09	1.11716892e−08	−3.89172071e−11
j	α_5 s	α_6 c	α_7 c	α_8 s
0		1.00090746e+00	−6.31406957e−02	
1	−2.19257075e+00	2.87092175e−04	3.88563862e+02	−3.89743726e+02
2	−3.33721049e−04	7.18717800e−03	1.73691020e−01	−1.73427917e−01
3	−3.29500143e−03	−2.35118315e−06	3.38290807e+00	−3.38569649e+00
4	−3.10063505e−07	4.58575897e−05	1.57483757e−04	−1.55588663e−04
5	−1.27777734e−05	−3.84868362e−08	2.93636049e−02	−2.93758267e−02
6	2.65280641e−09	3.27067750e−07	−1.22443455e−05	1.22585121e−05
7	−6.52847925e−08	−4.40696648e−10	2.53893543e−04	−2.53959689e−04
8	3.89172071e−11	2.45260066e−09	−2.27892904e−07	2.28002922e−07

where x_k, y_k, $z_k \in \mathbb{R}$, k denotes the multiindex (k_1, k_2) and

$$\gamma_1 = \exp(i(\omega_0 t + \varphi_0)) , \qquad \gamma_2 = \exp(i\omega_s t) ,$$

where ω_0 is a prefixed frequency. The phase φ_0 is arbitrary. Moreover, we have symmetry conditions $x_k = x_{-k}$, $y_k = -y_{-k}$ and $z_k = z_{-k}$.

The second order differential equations of the QBCP are

$$-\ddot{x} + 2\dot{y} + (1 + 2c_2)x + \sum_{n\geq 2} c_{n+1}(n+1)\, T_n + \alpha_{10}\ddot{x} + \alpha_{11}\dot{x} + \alpha_{12}\dot{y} +$$

$$+\alpha_{13}x + \alpha_{14}y + \alpha_{15} + \alpha_{16}\sum_{n\geq 2} c_{n+1}(n+1)\, T_n + \alpha_{20}\sum_{n\geq 2}\frac{\partial G_n}{\partial x} = 0 , \qquad (4)$$

$$-\ddot{y} - 2\dot{x} - (c_2 - 1)y + y\sum_{n\geq 2} c_{n+1}R_{n-1} + \alpha_{10}\ddot{y} - \alpha_{12}\dot{x} + \alpha_{11}\dot{y} -$$

$$-\alpha_{14}x + \alpha_{17}y + \alpha_{18} + \alpha_{16}y\sum_{n\geq 2} c_{n+1}R_{n-1} + \alpha_{20}\sum_{n\geq 2}\frac{\partial G_n}{\partial y} = 0 , \qquad (5)$$

$$-\ddot{z} - c_2 z + z\sum_{n\geq 2} c_{n+1}R_{n-1} + \alpha_{10}\ddot{z} + \alpha_{11}\dot{z} + \alpha_{19}z +$$

$$+\alpha_{16}z\sum_{n\geq 2} c_{n+1}R_{n-1} + \alpha_{20}\sum_{n\geq 2}\frac{\partial G_n}{\partial z} = 0 . \qquad (6)$$

The functions α_k for $k = 10, \ldots, 20$ are T-periodic and they can be computed from the previous α_k. The terms of the summations come from the expansion of the potential in Legendre polynomials. The coefficients c_n were introduced in [5] for the RTBP. T_n, R_n and G_n are polynomials in x, y and z, but G_n is also T-periodic on time. These functions are computed recursively (see [3] and [1]).

Let $\mathcal{F}$ be the space of real Fourier functions with two basic frequencies ω_0 and ω_s, $\mathcal{F}_0$ be the subspace of $\mathcal{F}$ of the even functions and $\mathcal{F}_1$ be the subspace of $\mathcal{F}$ of the odd functions. The equations 4, 5 and 6 allow to define

$$F : \mathcal{F}_0 \times \mathcal{F}_1 \times \mathcal{F}_0 \longrightarrow \mathcal{F}_0 \times \mathcal{F}_1 \times \mathcal{F}_0 \ .$$

So, the problem is reduced to solve a non linear equation $F(X) = 0$. It can be done using Newton's method

$$X^{(j+1)} = X^{(j)} - DF^{-1}(X^{(j)})F(X^{(j)}) \ ,$$

with $X^{(0)}$ the Fourier coefficients of a Halo orbit of the RTBP. As the equation is infinite dimensional, it is necessary a truncation and the use of an adaptative method. Let $M_j = \|F(X^{(j)})\|_\infty$ and $\kappa \in \mathbb{R}$, $0 < \kappa \ll 1$, say $\kappa = 10^{-5}$. In each iteration of the Newton's method, the selected frequencies are the ones such that their coefficients in $F(X^{(j)})$ are larger than κM_j. We remark that some steps of a continuation method are required.

The figure 2 shows the amplitude of the orbits computed. We realize the gap corresponding the low order resonance $\omega_0 = 2\omega_s$. The dynamics of this zone has not been studied yet.

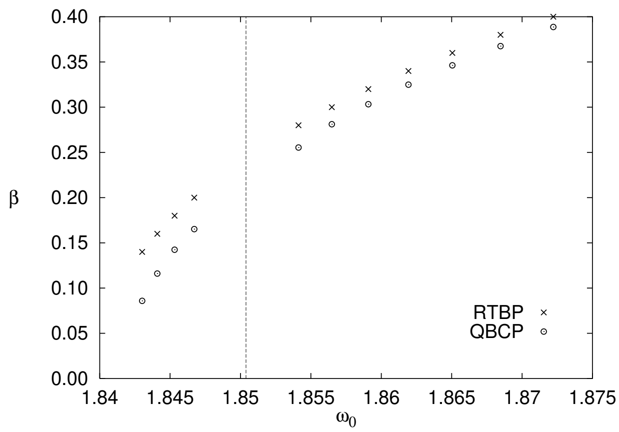

Figure 2. Frequency ω_0 and z-amplitude of the Halo orbits around L_2. The vertical dashed line corresponds to the frequency $\omega_0 = 2\omega_s$.

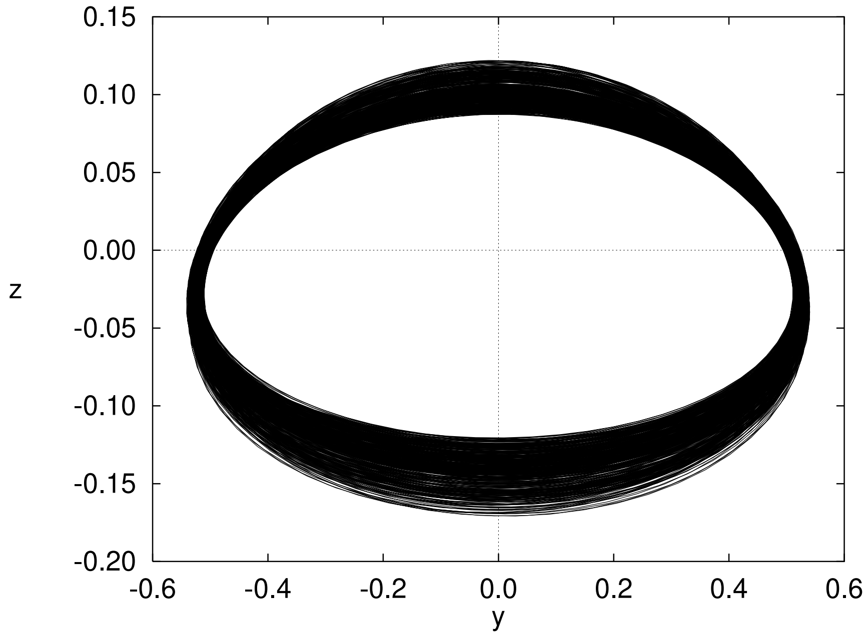

Figure 3. yz projection of a Halo orbit in the Solar System for a time span about 19 years. The unit of length is the distance between Moon and L_2 (about 60000 km).

4. Translunar Halo orbits in the real Solar System

The motivation to study the translunar Halo orbits is the application to spacecraft missions. These orbits were proposed by Farquhar in [2] as a means to stablish a continuous communications link between the Earth and the far side of the Moon. However, it was difficult to compute this kind of orbits for a long time interval. Presently, Halo orbits of the QBCP can be refined to Halo orbits of the Solar System for a long time span. The figure 3 shows one of these orbits for a time span of 19 years. To produce this orbit a Parallel Shooting method and JPL ephemeris are used (see [1]).

References

1. Andreu, M.A. (in preparation) *The Quasi-bicircular Problem*, Thesis, Dept. Matemàtica Aplicada i Anàlisi, Universitat de Barcelona.
2. Farquhar, R.W. (1971) *The Utilization of Halo Orbits in Advanced Lunar Operations*, NASA TN D-6365.
3. Gómez G., Jorba A., Masdemont J. and Simó C. (1991) *Study refinement of semi-analytical halo orbit theory*, ESOC contract 8625/89/D/MD(SC), Final Report.
4. Marchal, C. (1990) *The Three-Body Problem*, Elsevier.
5. Richardson, D.L. (1980) A note on a Lagragian formulation for motion about the collinear points, *Celestial Mechanics* **22**, 231–236.