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ON BIRKHOFF'S CONJECTURE ABOUT CONVEX BILLIARDS

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Abstract. Birkhoff conjectured that the elliptic billiard was the only integrable convex billiard. Here we prove a local version of this conjecture: *any* non-trivial symmetric entire perturbation of an elliptic billiard is non-integrable.

1. Introduction

Let us consider the problem of the "convex billiard table": let C be an (analytic) closed convex curve of the plane $\mathbb{R}^2$, parameterized by $\gamma : \mathbb{T} \longrightarrow C$, where $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, in such a way that C is traveled counterclockwise. Suppose that a material point moves inside C and collides with C according to the law "the angle of incidence is equal to the angle of reflection". Following Birkhoff [1], this discrete dynamical system can be modeled by an (analytic) a.p.m. in an annulus. More precisely, consider the annulus $\mathbb{A} = \{z = (\varphi, v) \in \mathbb{T} \times \mathbb{R} : |v| < |\dot\gamma(\varphi)|\}$, where the coordinate φ is the parameter on C and $v = |\dot\gamma(\varphi)| \cos\vartheta$, with $\vartheta \in (0, \pi)$ the angle of incidence-reflection of the material point. In this way, we obtain a map $T : \mathbb{A} \longrightarrow \mathbb{A}$ given by $(\varphi, v) \longmapsto (\Phi, V)$ that models the billiard (see Figure 1).

The function $\mathcal{S} : \{(\varphi, \Phi) \in \mathbb{T}^2 : \varphi \neq \Phi\} \longrightarrow \mathbb{R}$ defined by $\mathcal{S}(\varphi, \Phi) = |\gamma(\varphi) - \gamma(\Phi)|$ is a generating function of T, i.e.:

$$
\begin{aligned}
\frac{\partial \mathcal{S}}{\partial \varphi}(\varphi, \Phi) &= \frac{\langle \gamma(\varphi) - \gamma(\Phi), \dot\gamma(\varphi)\rangle}{|\gamma(\varphi) - \gamma(\Phi)|} = -\,|\dot\gamma(\varphi)| \cos\vartheta = -v,\\
\frac{\partial \mathcal{S}}{\partial \Phi}(\varphi, \Phi) &= \frac{\langle \gamma(\Phi) - \gamma(\varphi), \dot\gamma(\Phi)\rangle}{|\gamma(\varphi) - \gamma(\Phi)|} = |\dot\gamma(\Phi)| \cos\Theta = V,
\end{aligned}
$$

and consequently T is an a.p.m. ($\det DT(\varphi, v) = 1$), and (φ, v) are canonical conjugated coordinates.

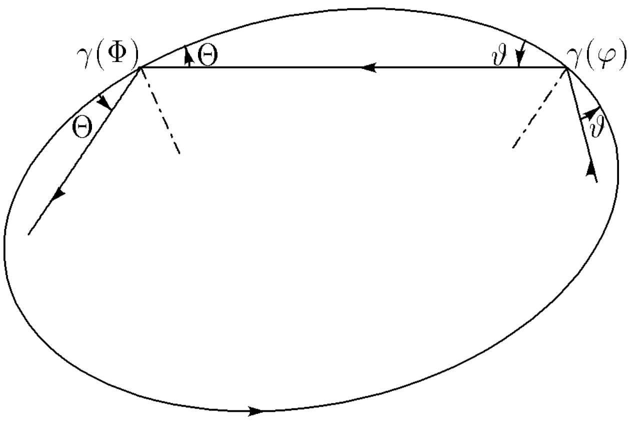

Figure 1: $T(\varphi, v) = (\Phi, V)$, where $v = |\dot{\gamma}(\varphi)| \cos \vartheta$ and $V = |\dot{\gamma}(\Phi)| \cos \Theta$.

Remark 1.1 The orbits of T (biinfinite sequences $(z_n)_{n\in\mathbb{Z}} \subset \mathbb{A}$ such that $T(z_n) = z_{n+1}$) are in one-to-one correspondence with the critical points of the formal series (called the *action*)

$$W((\varphi_n)_{n\in\mathbb{Z}}) = \sum_{n\in\mathbb{Z}} \mathcal{S}(\varphi_n, \varphi_{n+1}),$$

since these critical points $(\varphi_n)_{n\in\mathbb{Z}} \subset \mathbb{T}$ satisfy the equations

$$(1) \qquad \partial_1 \mathcal{S}(\varphi_n, \varphi_{n+1}) + \partial_2 \mathcal{S}(\varphi_{n-1}, \varphi_n) = 0, \qquad \forall n \in \mathbb{Z}.$$

Thus, having a generating function allows us to work with only half of the coordinates (the *base* coordinates, i.e., the φ's). The *fiber* coordinates (i.e., the v's) are superfluous.

Remark 1.2 Let C and C' two closed convex curves such that one is the image of the other by a similarity. Then the two associated a.p.m. T, T', have an equivalent dynamics since the angle of incidence-reflection remains unchanged by the similarity.

The map T has no fixed points but is geometrically clear that it has periodic orbits of period 2, corresponding to opposite points with the maximum and "minimum" distance between them. In these orbits the angle of incidence-reflection is $\pi/2$ and thus $v = 0$.

To study the dynamics of these 2-periodic points of T, it is better to consider them as fixed points of T^2, and study T^2. But since it is not easy to find the generating function for T^2, we instead introduce a simplification.

Suppose now that C is *symmetric* with regard to a point. By the previous remark 1.2, we can assume that this point is the origin. Then it is possible to choose a parameterization γ of C such that $\gamma(\varphi+\pi) = -\gamma(\varphi)$ and the 2-periodic orbits are of the form $(\varphi_0, 0)$, $(\varphi_0+\pi, 0)$, that is, two opposite points over C.

Remark 1.3 We can consider the variable φ defined modulus π in the symmetric case.

Let now $R : \mathbb{A} \longrightarrow \mathbb{A}$ be the *involution* $R(\varphi, v) = (\varphi+\pi, v)$ (R^2 is the identity). By the symmetry of C, T and R commute and it is a commonplace to use this symmetry to convert the 2-periodic points into fixed points. Concretely, we define a new map $F : \mathbb{A} \longrightarrow \mathbb{A}$ by $F = R \circ T$. Since $F^2 = T^2$, the dynamics of F and T are equivalent. Moreover, since $\gamma(\Phi+\pi) = -\gamma(\Phi)$, it is easy to check that

$$\mathcal{L}(\varphi, \Phi) = \mathcal{S}(\varphi, \Phi+\pi) = |\gamma(\varphi) + \gamma(\Phi)| \tag{2}$$

is a generating function for F, and consequently F is an a.p.m.

2. Elliptic Billiards

The simplest examples of convex curves are the ellipses. It is clear that the case of a circumference is very degenerated for a billiard, since it consists only of 2-periodic orbits. So, let us consider now a non-circular ellipse:

$$C_0 = \left\{ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \right\} = \{\gamma_0(\varphi) = (a\cos\varphi, b\sin\varphi) : \varphi \in \mathbb{T}\},$$

with $a^2 \neq b^2$. Without loss of generality we can assume that $a^2 - b^2 = 1$ (we change the ellipse using a similarity, if necessary). Thus $a > 1$, $b > 0$ and the foci of the ellipse are $(\pm 1, 0)$. Let us denote $T_0 : \mathbb{A} \longrightarrow \mathbb{A}$ the analytic a.p.m. associated to the ellipse C_0, and $F_0 = R \circ T_0$.

It is clear that the points $(0,0)$ and $(\pi,0)$ form a 2-periodic orbit for T_0 that corresponds to the vertexes $(\pm a, 0)$ of the ellipse, and hence $z_\infty := (0,0)$ is a fixed point for F_0. ($(\pi/2, 0)$ is another fixed point for F_0.)

Birkhoff was the first one in noticing that this system is *integrable*, i.e., there exists an analytic function $H(\varphi, v)$, called *first integral*, that preserves the orbits of T_0: $H \circ T_0 = H$. As a consequence, the curves $\{H = \text{constant}\}$ are invariant under T_0. Obviously, H is also a first integral of $F_0 = R \circ T_0$.

In fact, it is not difficult to check that $H(\varphi, v) = (\sin^2\varphi - v^2)/2$ is such a first integral, under the assumption $a^2 - b^2 = 1$. (See, for instance, [2] for the derivation of $I = -2H + 1$.)

In addition to the involution R, the ellipse C_0 has another symmetry $R^* : \mathbb{A} \longrightarrow \mathbb{A}$ given by $R^*(\varphi, v) = (\pi - \varphi, v)$. R^* is also an involution, and moreover $F_0^{-1} = R^* \circ F_0 \circ R^*$, i.e., F_0 is R^*-reversible.

The dynamics of F_0, based in the level curves of H is drawn in Figure 2 where the resemblance with the phase portrait of a pendulum shows up clearly.

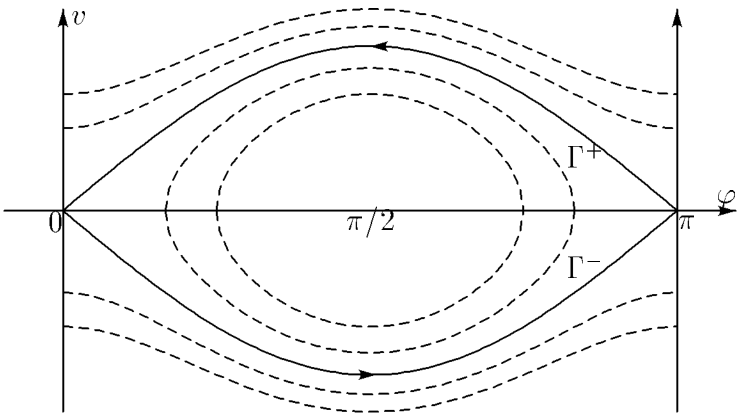

Figure 2: Phase portrait of F_0; $\Gamma^\pm$ are the separatrices of F_0.

The main properties of F_0 are listed in the following Lemma, whose proof can be found in [3].

Lemma 2.1 *a)* $z_\infty = (0,0)$ *is a saddle fixed point of* F_0 *and* $Spec[DF_0(z_\infty)] = \{\lambda, \lambda^{-1}\}$, *with* $\lambda = (a+1)(a-1)^{-1} > 1$.

Moreover, if $h := \ln\lambda$ *the following expressions hold*

$$a = \coth(h/2), \quad b = \operatorname{cosech}(h/2). \tag{3}$$

b) $\Gamma^{\pm} = \{(\varphi, \pm\sin\varphi) : 0 < \varphi < \pi\}$ *are the* separatrices *of* F_0*: if* $z \in \Gamma^{\pm}$*, then* $F_0^n(z) \longrightarrow z_\infty$ *when* $n \to \pm\infty$*.*

c) If $\sigma^{\pm}(t) = (\varphi(\pm t), \pm v(t))$*, where* $\varphi(t) = \arccos(\tanh t) = \arcsin(\operatorname{sech} t)$ *and* $v(t) = \operatorname{sech} t$*, then* $F_0(\sigma^{\pm}(t)) = \sigma^{\pm}(t+h)$*. In other words,* $\sigma^{\pm}$ *are* natural parameterizations *of* $\Gamma^{\pm}$ *(with regard to* F_0*).*

d) Let $\Phi(t) = \varphi(t+h)$*. Then*

$$b\frac{\sin\varphi(t) + \sin\Phi(t)}{|\gamma_0(\varphi(t)) + \gamma_0(\Phi(t))|} = \operatorname{sech}(t + h/2). \tag{4}$$

2.1. Entire elliptic billiard perturbations

Birkhoff conjectured that the elliptic billiard was the only integrable convex billiard. Our goal is to see that this is locally true for the symmetric billiards, i.e., any non-trivial symmetric entire perturbation is non-integrable. (Non-trivial perturbation means not reducible to an ellipse.)

Let $\{C_\varepsilon\}$ be an arbitrary family of perturbations of the ellipse C_0, consisting of analytic curves depending on a $\mathcal{C}^2$ way on ε and symmetric with regard to a point O_ε. Let us denote by $Q_\varepsilon^{\pm}$ the two furthest (and opposite) points over C_ε with $Q_0^{\pm} = (\pm a, 0)$. Using a similarity that takes O_ε and $Q_\varepsilon^{\pm}$ to $(0,0)$ and $(\pm a, 0)$ respectively, the initial family can be put in the following form

$$C'_\varepsilon = \left\{(x,y) \in \mathbb{R}^2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \varepsilon P(x,y,\varepsilon) = 1\right\}, \tag{5}$$

where:

I) P is analytic in x, y and at least $\mathcal{C}^1$ in ε,

II) $P(x,y,\varepsilon) = P(-x,-y,\varepsilon)$,

III) $P(a,0,\varepsilon) = \partial_y P(a,0,\varepsilon) = 0$, or equivalently, like

$$(6) \qquad C'_\varepsilon = \{\gamma(\varphi,\varepsilon) = (a\cos\varphi, \sin\varphi\,[b+\varepsilon\eta(\varphi,\varepsilon)]) : \varphi \in \mathbb{T}\},$$

where:

i) η is analytic in φ and at least $\mathcal{C}^1$ in ε,

ii) η is π-periodic in φ.

From III), it follows that

$$P(a\cos\varphi, b\sin\varphi, \varepsilon) = p(a\cos\varphi, b\sin\varphi, \varepsilon)\sin^2\varphi,$$

with p satisfying also I), II). It is easy to check that, in first order in ε, the relation between η and P is given by

$$(7) \qquad P(a\cos\varphi, b\sin\varphi, 0) = -2b\eta(\varphi, 0)\sin^2\varphi.$$

Thus, if $P(\cdot,\cdot,0)$ is an entire function in the variables x and y, the same happens to $\eta_1 := \eta(\cdot, 0)$. It is clear that if $\eta_1 =$ constant, C'_ε is, in first order, a family of ellipses.

Definition 2.1 *Let $\{C_\varepsilon\}$ be a perturbation of the ellipse C_0. We say that $\{C_\varepsilon\}$ is a* non-trivial symmetric entire perturbation *of the ellipse when it can be put, using similarities, in the form (6) and moreover, $\eta_1 := \eta(\cdot, 0)$ is a non-constant entire function.*

Let T_ε be the map in the annulus associated to the billiard in C_ε, where the perturbation considered is a symmetric entire one and let $F_\varepsilon = R \circ T_\varepsilon$. If ε is small enough, C_ε is an analytic convex closed curve, and thus $\{F_\varepsilon\}_{|\varepsilon|\ll 1}$ is a family of analytic a.p.m. with generating function $\mathcal{L}_\varepsilon(\varphi,\Phi,\varepsilon) = |\gamma(\varphi,\varepsilon)+\gamma(\Phi,\varepsilon)|$ that can be written as $\mathcal{L}_\varepsilon(\varphi,\Phi,\varepsilon) = \mathcal{L}_0(\varphi,\Phi) + \varepsilon\mathcal{L}_1(\varphi,\Phi) + \mathrm{O}(\varepsilon^2)$, where

$$(8) \qquad \begin{aligned} \mathcal{L}_0(\varphi,\Phi) &= |\gamma_0(\varphi,\varepsilon)+\gamma_0(\Phi,\varepsilon)|, \\ \mathcal{L}_1(\varphi,\Phi) &= b\frac{\sin\varphi+\sin\Phi}{|\gamma_0(\varphi)+\gamma_0(\Phi)|}[\sin\varphi\,\eta_1(\varphi)+\sin\Phi\,\eta_1(\Phi)]. \end{aligned}$$

For ε small enough, the origin z_∞ is again a hyperbolic point of F_ε. This point z_∞ lies in the intersection of the invariant curves $\mathcal{W}^{\mathrm{s}}_\varepsilon$, $\mathcal{W}^{\mathrm{u}}_\varepsilon$, such that any $(z_n)_{n\in\mathbb{Z}}$ orbit in the manifold $\mathcal{W}^{\mathrm{s,u}}_\varepsilon$ tends to

z_∞ at an exponential rate as $n \to +\infty, -\infty$. These invariant curves will not longer coincide for $\varepsilon \neq 0$. But for ε small enough, there exist open sets $\mathcal{U}^{\mathrm{s,u}}$ in $\mathcal{W}^{\mathrm{s,u}}_\varepsilon$ and analytic functions $\Psi^{\mathrm{s,u}}_{\varepsilon,*} : \mathcal{I}^{\mathrm{s,u}} \to \mathbb{R}$ such that $\mathcal{U}^{\mathrm{s,u}} = \mathrm{Graph}(\Psi^{\mathrm{s,u}}_{\varepsilon,*})$, with $\mathcal{I} := \mathcal{I}^{\mathrm{s}} \cap \mathcal{I}^{\mathrm{u}}$ of size a little bit less than π. Let us see that the primitives $\psi^{\mathrm{s,u}}_{\varepsilon,*}$ of the functions $\Psi^{\mathrm{s,u}}_{\varepsilon,*}$ have a nice interpretation, crucial to the derivation of the Melnikov potential.

Proposition 2.1 *Let $\psi^{\mathrm{s,u}}_\varepsilon : \mathcal{W}^{\mathrm{s,u}}_\varepsilon \to \mathbb{R}$ be the analytic functions defined by*

$$\psi^{\mathrm{s}}_\varepsilon(z^{\mathrm{s}}) = -\sum_{n\geq 0} \mathcal{L}_\varepsilon(\varphi^{\mathrm{s}}_n, \varphi^{\mathrm{s}}_{n+1}), \quad \psi^{\mathrm{u}}_\varepsilon(z^{\mathrm{u}}) = \sum_{n<0} \mathcal{L}_\varepsilon(\varphi^{\mathrm{u}}_n, \varphi^{\mathrm{u}}_{n+1}), \tag{9}$$

where, if $z^{\mathrm{s,u}} \in \mathcal{W}^{\mathrm{s,u}}_\varepsilon$, $z^{\mathrm{s,u}}_n = (\varphi^{\mathrm{s,u}}_n, v^{\mathrm{s,u}}_n) = F^n_\varepsilon(z^{\mathrm{s,u}})$ for $n \in \mathbb{Z}$. Then:

1. $\psi^{\mathrm{s,u}}_\varepsilon(z_\infty) = 0$.
2. $\psi^{\mathrm{s,u}}_{\varepsilon,*}(\varphi) = \psi^{\mathrm{s,u}}_\varepsilon(z^{\mathrm{s,u}}), \forall \varphi \in \mathcal{I}^{\mathrm{s,u}}$, $z^{\mathrm{s,u}} = (\varphi, \Psi^{\mathrm{s,u}}_{\varepsilon,*}(\varphi))$.

This result is a direct consequence of the variational principle (1). Moreover, the properties *1.* and *2.* determine completely the functions $\psi^{\mathrm{s,u}}_\varepsilon$.

To measure the separation between invariant curves, we will use the *Melnikov potential*, i.e., a primitive of the Melnikov function, that is nothing else that the *action* $\sum \mathcal{L}(\varphi_n, \varphi_{n+1}, \varepsilon)$ avaluated on the unperturbed separatrices $\Gamma := \Gamma^+ \cup \Gamma^-$ up to the first order in ε.

Proposition 2.2 *Let $L : \Gamma \to \mathbb{R}$ be the function defined by*

$$L(z) = \sum_{n\in\mathbb{Z}} \mathcal{L}_1(\varphi_n, \varphi_{n+1}), \qquad z_n = (\varphi_n, v_n) = F^n_0(z). \tag{10}$$

Then:

(i) L is well-defined (the sum in (10) is absolutely convergent), analytic and F_0-invariant (i.e., $L \circ F_0 = L$).

(ii) If L is not identically constant and ε is small enough, the invariant curves $\mathcal{W}^{\mathrm{u}}_\varepsilon$, $\mathcal{W}^{\mathrm{s}}_\varepsilon$ intersect along a finite contact homoclinic *orbit of F_ε, and F_ε is non-integrable.*

Proof of the Proposition. We only sketch here the proof. For more details, see [4].

(i) The sum is absolutely convergent since any orbit in the separatrix Γ tends to z_∞ at an exponential rate as $|n| \to \infty$ and $\mathcal{L}_1(0,0) = 0$. Thus L is well-defined and analytic. Moreover, a shift in the index of the sum does not changes its value, so the F_0-invariance of L is proved.

(ii) Our measure of the distance between the perturbed asymptotic manifolds is the difference of fiber coordinates (the v's)

$$\Delta_*(\varphi, \varepsilon) := \frac{\mathrm{d}\psi^{\mathrm{u}}_{\varepsilon,*}(\varphi)}{\mathrm{d}\varphi} - \frac{\mathrm{d}\psi^{\mathrm{s}}_{\varepsilon,*}(\varphi)}{\mathrm{d}\varphi} = \varepsilon M_*(\varphi) + \mathrm{O}(\varepsilon^2).$$

M_* gives in first order in ε the distance between invariant curves, and thus can be considered as a Melnikov function.

For a primitive δ_* of Δ_* with respect to φ, $\Delta_* := \mathrm{d}\delta_*/\mathrm{d}\varphi$, we have the following expression:

$$\delta_*(\varphi, \varepsilon) := \psi^{\mathrm{u}}_{\varepsilon,*}(\varphi) - \psi^{\mathrm{s}}_{\varepsilon,*}(\varphi) = \sum_{n\in\mathbb{Z}} \mathcal{L}_\varepsilon \left(\varphi_n^{\alpha(n)}, \varphi_{n+1}^{\alpha(n)}\right),$$

where $(\varphi_n^{\mathrm{s,u}}, v_n^{\mathrm{s,u}}) = F_\varepsilon^n(\varphi, \mathrm{d}\psi^{\mathrm{s,u}}_{\varepsilon,*}(\varphi)/\mathrm{d}\varphi)$, $\alpha(n) = \mathrm{s}$, if $n \geq 0$ and $\alpha(n) = \mathrm{u}$, if $n < 0$.

Let $L_* : [0,\pi] \to \mathbb{R}$ the analytic function defined by $L_*(\varphi) := L(\varphi, \mathrm{d}\psi_0(\varphi)/\mathrm{d}\varphi)$, where $\psi_0(\varphi) = -\cos\varphi$. Using the uniform approximations of $\mathcal{W}_\varepsilon^{\mathrm{s,u}}$ by Γ and the variational principle, it turns out that

$$\delta_*(\varphi, \varepsilon) = \varepsilon L_*(\varphi) + \mathrm{O}(\varepsilon^2).$$

Thus, $M_* = \mathrm{d}L_*/\mathrm{d}\varphi$, and now it is clear that if L is not identically constant, the same will happen to L_*. Finally, one can see that M_* is not zero, analytic, and has a zero of finite order, using the fact that $L_* \circ \varphi$ is h-periodic, where $\sigma(t) = (\varphi(t), v(t))$. Consequently, for ε small enough, the invariant curves $\mathcal{W}_\varepsilon^{\mathrm{u}}$, $\mathcal{W}_\varepsilon^{\mathrm{s}}$ intersect along a finite contact *homoclinic* orbit of F_ε. By a result of Cushman [5], F_ε is non-integrable. □

From now on, we consider only Γ^+. Using the parameterization provided by (2.1), we can write $\Gamma^+ = \{\sigma^+(t) = (\varphi(t), v(t))\}$, and consider the Melnikov potential

$$L(t) = \sum_{n\in\mathbb{Z}} f(t + hn), \text{ with } f(t) = \mathcal{L}_1(\varphi(t), \varphi(t+h)),$$

where for simplicity of notation, we write L instead of $L_* \circ \sigma$, with $\mathcal{L}_1$ as given in equation (8). Using formula (4), we arrive at an equivalent expression for f:

$$(11)\quad f(t) = \operatorname{sech}(t+\frac{h}{2})[\operatorname{sech}(t)\eta_1(\varphi(t)) + \operatorname{sech}(t+h)\eta_1(\varphi(t+h))].$$

2.2. Non-integrable billiards

The aim of this paper is to prove the following result.

Theorem 2.1 *Let $\{C_\varepsilon\}$ be any non-trivial symmetric entire perturbation of an ellipse. Then the billiard in C_ε is non-integrable for $0 < |\varepsilon| \ll 1$.*

Proof of the Theorem. By proposition 2.2, we only have to prove that L is non constant. To this end, we will use the following result.

Lemma 2.2 *In a complex neighbourhood D of $t = \pi\mathrm{i}/2$, there exists a analytic function G such that $L(t)$ reads as*

$$L(t) = \frac{2a}{b}\operatorname{sech}(t+h/2)\operatorname{sech}(t-h/2)\eta_1(\varphi(t)) + G(t).$$

Accepting for the moment this lemma, and using the fact that $\operatorname{sech}(t+h/2)\operatorname{sech}(t-h/2)$ is analytic non zero on $t = \pi\mathrm{i}/2$, it turns out that $t = \pi\mathrm{i}/2$ is a singular point of $L(t)$ if and only if the same happens to $\eta_1(\varphi(t))$. But by hypothesis, η_1 is a π-periodic entire function, and by lemma 2.1, $\sin\varphi(t) = \operatorname{sech}(t)$ and $\cos\varphi(t) = \tanh(t)$ have simple poles at $t = \pi\mathrm{i}/2$ and no more singularities on $\Im t = \pi/2$. Since η_1 is non constant, $t = \pi\mathrm{i}/2$ is a singular point of $\eta_1(\varphi(t))$ and consequently $L(t)$ is not a constant. □

Proof of Lemma 2.2. Looking at the expression (11) of f, and using the fact that η_1 is entire, the only possible singularities of f with $\Im t = \pi/2$ are $t = \pi\mathrm{i}/2$, $t = \pi\mathrm{i}/2 - h$, and $t = \pi\mathrm{i}/2 - h/2$. Nevertheless, $t = \pi\mathrm{i}/2 - h/2$ is a removable singularity, since it is a simple pole of $\operatorname{sech}(t+h/2)$, and also a zero of $\operatorname{sech}(t)\eta_1(\varphi(t)) + \operatorname{sech}(t+h)\eta_1(\varphi(t+h))$. So we can write

$$L(t) = \sum_{n\in\mathbb{Z}} f(t+nh) = f(t-h) + f(t) + G_1(t)$$

where $G_1(t) = \sum_{n \neq 0,1} f(t + nh)$ is analytic on $t = \pi\mathrm{i}/2$, and

$$\begin{aligned} f(t-h) + f(t) = & \\ & \operatorname{sech}(t-h/2)\left[\operatorname{sech}(t-h)\eta_1(\varphi(t-h)) + \operatorname{sech}(t)\eta_1(\varphi(t))\right] \\ & + \operatorname{sech}(t+h/2)\left[\operatorname{sech}(t)\eta_1(\varphi(t)) + \operatorname{sech}(t+h)\eta_1(\varphi(t+h))\right] \\ = \; & (\operatorname{sech}(t-h/2) + \operatorname{sech}(t+h/2))\operatorname{sech}(t)\eta_1(\varphi(t)) + G_2(t), \end{aligned}$$

where $G_2(t) = \operatorname{sech}(t-h/2)\operatorname{sech}(t-h)\eta_1(\varphi(t-h)) + \operatorname{sech}(t+h/2) \times \operatorname{sech}(t+h)\eta_1(\varphi(t+h))$ is analytic on $t = \pi\mathrm{i}/2$.

And finally, using the formula $\operatorname{sech}(t-h/2) + \operatorname{sech}(t+h/2) = 2\cosh(h/2)\operatorname{sech}(t-h/2)\operatorname{sech}(t+h/2)\cosh(t)$, and relations (3), the lemma follows with $G(t) = G_1(t) + G_2(t)$. □

Remark 2.1 The same proof works for the point $t = \frac{3\pi}{2}\mathrm{i}$ (instead of $t = \frac{\pi}{2}\mathrm{i}$). The assumption of entire function on η_1 has only been used to ensure that for $t_p = \frac{\pi}{2}\mathrm{i}$ or $t_p = \frac{3\pi}{2}\mathrm{i}$, $\eta_1(\varphi(t))$ has an isolated singular point at t_p but is analytic on $t_p + nh$ for $n \neq 0$.

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