THE USE OF TIME AS A COMPLEX VARIABLE[1]

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Abstract. Dynamical systems try to describe the behaviour of mathematical models of the natural sciences. These models are, in the deterministic case, ordinary differential equations, diffeomorphisms, partial differential equations, differential delay equations, etc. Being all of them evolution equations, to model the behaviour of the natural phenomena, time appears as the independent variable. In this paper we review the importance of considering time as a complex variable instead of a real one, as it would be suggested by the physical meaning of time. The topics discussed include bounds on the averaging theory for slow systems, splitting of separatrices, quasiperiodic equations and solutions, integrability of Hamiltonian systems, delay in the loss of stability, complex homoclinic points, etc. In all cases we assume that the models are analytic.

Resum. Els sistemes dinàmics intenten descriure el comportament de models matemàtics de les ciències de la natura. Aquests models són, en el cas determinista, equacions diferencials ordinàries, difeomorfismes, equacions en derivades parcials, equacions diferencials amb retard, etc. Essent totes elles equacions d'evol·lució, per a modelar els fenòmens naturals, el temps apareix com a variable independent. En aquest article passem revista a l'importància de considerar el temps com a variable complexa en lloc de real, com suggeriria el significat físic del temps. Els temes discutits inclouent fites de la teoria de les mitjanes per a sistemes lents, trencament de separatrius, equacions i solucions quasiperiòdiques, integrabilitat de sistemes hamiltonians, retard en la pérdua d'estabilitat, punts homoclínics complexos, etc. En tots els casos suposem que els models són analítics.

Resumen. Los sistemas dinámicos intentan describir el comportamiento de modelos matemáticos de las ciencias de la naturaleza. Estos modelos son, en el caso determinista, ecuaciones diferenciales ordinarias, difeomorfismos, ecuaciones en derivadas parciales, ecuaciones diferenciales con retraso, etc. Siendo todas ellas ecuaciones de evolución, para modelizar los fenómenos naturales, el tiempo aparece como variable independiente. En este artículo pasamos revista a la importancia de considerar el tiempo como variable compleja en lugar de real, como sugeriría el significado físico del tiempo. Los temas discutidos incluyen cotas de la teoría de promedios para sistemas lentos, rotura de separatrices, ecuaciones y soluciones quasiperiódicas, integrabilidad de sistemas hamiltonianos, retraso en la pérdida de estabilidad, puntos homoclínicos complejos, etc. En todos los casos suponemos que los modelos son analíticos.

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1 Introduction

Time has been considered as a paradigm of a real variable. If we set some origin, $t = 0$, at the present moment, then the future corresponds to $t > 0$. Furthermore, there is a well defined order in time: the past precedes the future.

Let us consider a mathematical model of a natural phenomenon. If some magnitudes characterizing the phenomenon are denoted by x, belonging to some open set, Ω, in $\mathbb{R}^n$, a typical model for the evolution of x with time are ordinary differential equations

$$\frac{dx}{dt} = f\,(t, x), \tag{1}$$

which, together with the state of the system, $x\,(t_0) = x_0$, at some specific value of time, t_0, determine x as a function of t, $x = \varphi\,(t; t_0, x_0)$. This function is defined while (1) makes sense, that is, while $(t, \varphi(t;\, t_0, x_0))$ belongs to the domain where f is defined. In what follows we shall always assume, unless we say something against, that the function f is analytic in all its variables. Then φ is also analytic in all its variables.

I am not aware of any physical phenomenon allowing to "travel with t complex". Perhaps this is due to physical or human limitations. However, from the mathematical point of view, it is possible to consider the variables (t, x) in (1) as complex variables. Then the solutions, $\varphi\,(t;\, t_0, x_0)$, are analytical and can be continuated to complex values through its analyticity domain. This domain can be (usually it is) rather involved, the function φ having singularities for several values $t = t_s$ when we fix (t_0, x_0).

We can consider a path, Γ, in the complex numbers $\mathbb{C}$ contained in the analyticity domain of φ. We can not only "travel to the past or to the future" through the complexes, but also return to the same starting point, t_0. If Γ is a closed path and the connex component of its complement in $\mathbb{C}$ contains no singularities nor branching points, then the final state will coincide with x_0. But in the presence of singularities or branching points we can return to t_0 and have a state different from x_0. Even if we return to the same x_0 the following can happen: Assume x_0 is a regular point, that is, $f\,(t, x_0)$ is not identically zero. Then the flow box theorem assures the existence of $n-1$ local first integrals if f is autonomous (f is independent of t). These local integrals can be continuated analytically along Γ. But when returning to x_0 these integrals can be different from the ones defined initially. This will prevent the existence of global integrals of the motion, and this fact is certainly related to non predictable behaviour.

To consider (1) and its solutions with complex variables is not only a mathematical game. Indeed, there are phenomena "difficult to see in the reals" which become clear in the complexes. The most elementary one appears in the case of the roots of a polynomial. A polynomial with real coefficients and having all the roots real can be considered as depending on parameters. When these are changed we can have multiple roots and a further variation of the parameters leads, generically, to roots moving to the complexes. Going back to (1) we can have fixed points, periodic solutions, etc. which disappear from "the real world" at some bifurcation value of a parameter. Allowing for complex time and variables we can continue them easily. Eventually, a new variation of the parameters can produce that these objects "land again on the reals". So, disappearitions and reappearitions of different objects can be better explained by following them across the complexes.

From the technical point of view the complex time allows for bounds on the solutions or in the transformations which can be done to (1) which are simpler that just using real variables, even if we are interested only on the "real" results.

Some phenomena are rather subtle in the real variables. I am refering to the so called "exponentially small things". We can have an equation like (1) but depending on some small parameter ε:

$$\frac{dx}{dt} = f(t,x) + \varepsilon f_1(t,x). \tag{2}$$

One can expect that the properties of the solutions of (2) will differ from the ones of (1) by $O(\varepsilon)$ or, eventually, by $O(\varepsilon^k)$ for some natural number k. This is the basis of the perturbation theory. One tries to obtain solutions of (2) by expanding in powers of ε. However it turns out that there are effects of ε which are "beyond all orders", that is, they are less than ε^k for all k, as large as you want. Typically they are of the form $\exp(-\frac{c}{\varepsilon^b})$, for some positive constants c and b, and they can not be detected by a "classical perturbation theory". They become transparent when complex variables are used.

Several objects, like invariant tori of (1), can be analytic and to have a sudden disappearition at some critical value of a real parameter, ε_c. Again this can be understood using complex variables to see the behaviour of the singularities of the representation of these objets when ε goes to ε_c. Typically, the domain of analyticity contains a strip around the real axis and the width of the strip goes to zero when ε goes to ε_c, because the singularities tend to the real domain.

In what follows I shall sketch several of these phenomena, trying to make clear the extremely important role of complex time.

2 Averaging of systems with slow dynamics

Let us consider variables which evolve slowly with time

$$\frac{dx}{dt} = \varepsilon f(t,x), \tag{3}$$

where ε is a real small parameter. By introducing a new time $s = \varepsilon t$, (3) is equivalent to

$$\frac{dx}{ds} = f\left(\frac{s}{\varepsilon}, x\right), \tag{4}$$

that is, a system under an excitation depending on a fast way on s, the faster the smaller ε is. If f is T-periodic with respect to t (and, hence, εT periodic in s) we can "average" f with respect to s:

$$\bar{f}(x) = \frac{1}{T\varepsilon} \int_0^{T\varepsilon} f\left(\frac{s}{\varepsilon}, x\right) ds = \frac{1}{T} \int_0^T f(t,x)\, dt\,, \tag{5}$$

and consider the "averaged system", obtained by replacing f by $\bar{f}$, as defined by (5), in (3) or (4):

$$\frac{dx}{dt} = \varepsilon \, \bar{f}\,(x). \tag{6}$$

The natural question is: What is the difference between the solutions of (3) and the ones of (6)? We can also ask if there are better methods to average.

To answer these questions we shall consider a more general problem. Instead of assuming f periodic in t we shall assume it quasiperiodic. An analytic quasiperiodic function, $g\,(t)$, is a function defined by

$$g\,(t) = F\,(\theta_1, \theta_2, \ldots, \theta_m), \tag{7}$$

where $\theta_j = \omega_j\, t + \alpha_j,\ j = 1, \ldots, m$, such that F is 2π periodic and analytic in all the variables θ. We say that F is analytic in a strip of width ρ if F is analytic for $|Im\,\theta_j| \leq \rho,\ j = 1, \ldots, m$. The constants ω_j and α_j are denoted as frequencies and phases, respectively. We assume that the frequencies $\omega_1, \ldots, \omega_m$ are independent over $\mathbb{Q}$, that is, any relation of the form $\sum_{j=1}^{m} k_j\, \omega_j = 0$ forces $k_1 = \ldots = k_m = 0$ if the coefficients k_j are integers. Otherwise some of the frequencies would be then linear combinations of the others with rational coefficients and, hence, not essential. Then we say that the function g in (7) is quasiperiodic with m basic frequencies.

If there is a vector of integers, $k_1, \ldots, k_m$, not all of them equal to zero and such that $\sum_{j=1}^{m} k_j\, \omega_j = 0$, we say that the frequencies are "resonant". To be basic means that there are no resonances. We want not only to have no resonances among the "external" (i.e., appearing in the time dependence) frequencies, but also to be not "too close" to resonance. We say that the basic frequencies satisfy a (c, τ) diophantine condition (DC for shortness) if

$$\left| \sum_{j=1}^{m} k_j\, \omega_j \right| \geq \frac{c}{|k|^{\tau}} \quad , \quad \forall\, k \in \mathbb{Z}^m \backslash \{0\}, \tag{DC}$$

where c and τ are positive numbers and $|k| = \sum_{j=1}^{m} |k_j|$.

Given a compact domain, D, in $\mathbb{R}^n$ or $\mathbb{C}^n$ we denote by $D + \rho$ its ρ-neighbourhood, that is $D + \rho = \{z \in \mathbb{C}^n \,|\, \exists\, y \in D,\ d\,(y, z) \leq \rho\}$. The symbol $\|\ \|$ will denote the supremum norm.

Theorem 1 [19]. *Assume that the equation (3) is analytic in x for $x \in D + \rho$, D compact in $\mathbb{R}^n$ or $\mathbb{C}^n$, and it is quasiperiodic analytic on a strip of width ρ with respect to t, with the basic frequencies satisfying a (DC) condition. Then there is $\varepsilon_0 > 0$ such that for all ε with $|\varepsilon| < \varepsilon_0$ there is an analytic change of coordinates, $x = h\,(t, z, \varepsilon)$, defined for $z \in D + \rho/2$ and quasiperiodic analytic on a strip of width $\rho/2$ with respect to t having the same basic frequencies than f, such that (3) is expressed in the variable z as*

$$\frac{dz}{dt} = \varepsilon\, g(z) + r\,(t, z, \varepsilon), \tag{8}$$

with $\|g\| < 2\,\|f\|$ and $\|r\| < \exp\,(\,-\,c_1/\varepsilon^{1/(\tau+1)})$, for some positive constant c_1. The bound on the remainder, r, is sharp, that is, it can not be improved in general.

This answers one of the questions posed before. The "averaged" system $\frac{dz}{dt} = \varepsilon\, g(z)$ differs from the initial one by an amount which is exponentially small in ε. Then the difference between the solutions of (8) (equivalent to (3)) and the averaged system can be bounded by using, for instance, Gronwall's lemma.

The other question, about better methods to average, is answered by the proof of theorem 1. One performs a sequence of transformations to cancel the time dependence. In the periodic case the step (5) would be just the first one. The full proof is obtained by combining Cauchy estimates (and here complex time is essential) of the solutions of partial differential equations leading to the (finite) sequence of changes of variables, with suitable choices of the different domains and some properties of arithmetic-geometric character of the "small divisors" $\sum_{j=1}^{m} k_j\, \omega_j$ appearing in these solutions.

Let me remark that the periodic case was proved by Neishtadt [14] and in that case one has $\tau = 0$ and no need of analyticity with respect to t. Just continuity is enough. Both in the periodic and in the quasiperiodic cases the number of transformations to be done on (3) to obtain the form (8) depends on ε. It is of the form $E\,[d/|\varepsilon|]$, where d is a positive constant and E denotes the integer part.

As exponentially small quantities appear in many other places in this paper, let we make a comment on them. Consider the function $\varepsilon \longmapsto \exp(-1/\varepsilon)$. The coefficient on the numerator of the exponential and the exponent of ε have been taken equal to 1 for simplicity. For $\varepsilon = 1, 10^{-1}, 100^{-1}$, the corresponding values are close to $0.37, 4.5 \cdot 10^{-5}$ and $3.7 \cdot 10^{-44}$, that is, by decreasing ε the function $\exp(-1/\varepsilon)$ decreases extremely fast. If, on the other hand, we assume that we can detect (or measure) things as small as 10^{-10} then the critical value $\varepsilon_c = 0.0434$ gives $\exp(-1/\varepsilon_c) \simeq 10^{-10}$, but just doubling or halving ε gives $\exp\,(-1/(2\,\varepsilon_c)) \simeq 10^{-5}$, a comparatively large magnitude, or $\exp\,(-1/(\varepsilon_c/2)) \simeq 10^{-20}$, completely negligible. Relative small variations in ε give very sharp variations in $\exp(-1/\varepsilon)$.

Assume the averaged system $\dot{z} = \varepsilon\, g(z)$ behaves in a smooth way, that is, the variational equations along a solution do not increase the norm quickly. This would be the case if we have an oscillator as averaged system. Assume also that we are satisfied with an agreement $O(10^{-6})$ between the exact and the averaged solutions. It follows that a good agreement would be found if $\varepsilon = 0.0434$, in the hypothesis given above, for values of t up to 10^4. But for $\varepsilon = 0.0217$ the agreement would be good even for $t = 10^{16}$. Probably long before that time the model would be no longer valid!

In the case of equations as (3) but with arbitrary dependence with respect to t (that is, nor periodic neither quasiperiodic, but still analytic) the use of complex time allows to produce bounds on the variation of some magnitudes. This is related to the adiabatic invariance, but we shall not discuss it here.

3 The splitting of separatrices

Consider one of the most elementary models: the mathematical pendulum. The related Hamiltonian is

$$H\,(q,p) = \frac{p^2}{2} - \cos\, q, \tag{9}$$

and from (9) we derive the equations of motion

$$\dot{q} = p, \quad \dot{p} = -\sin q, \quad \text{or} \quad \ddot{q} = -\sin q. \tag{10}$$

Of course q is defined modulus 2π. From (10) we obtain the fixed points $p = 0$ and $q = 0$ (stable) or $q = \pi$ (unstable). The unstable equilibrium corresponds to the pendulum having his mass at the top. There are motions leading to (or escaping from) this unstable equilibrium in infinite time. If the pendulum is at rest on the stable equilibrium and you kick it to have a speed $p < 2$ it will oscillate around the equilibrium solution (if p is close to 2, with a large amplitude). If $p > 2$ then it will make revolutions, but if you succeed in kicking with $p = 2$ it will move towards the unstable equilibrium. Of course, no friction is assumed in all the cases. The orbits ending at the unstable fixed point are called separatrices, and they also coincide with the orbits going to that point for $t \to -\infty$. The name is clear: They "separate" oscillations from revolutions. The separatrices lie on the level $H = 2$ of (9).

Now assume that you modify (10) by adding some very fast time periodic perturbation

$$\ddot{q} = -\sin q + \varepsilon^r \sin(t/\varepsilon), \tag{11}$$

where the exponent r measures the size of the perturbation, but can be even negative ($r \geq -2$) having in that case a very large perturbation if ε is small, and what we will say is still true [3], [6], [7]. The fixed points of (10) are now replaced by periodic solutions of (11), stable and unstable, respectively, of period $2\pi\varepsilon$.

The separatrices of (10) are now replaced by the so called stable and unstable manifolds of the unstable periodic orbit. The stable manifold, W^s_{po}, is the set of points going to the periodic orbit when $t \to +\infty$, and the unstable manifold, W^u_{po}, the ones going to the periodic orbit for $t \to -\infty$. But, in constrast with the system (10) these two separatrices do not coincide now. This is the so called phenomenon of "separatrices splitting". This is intimately related to the existence of non predictable motion.

However an application of the results on averaging gives that the splitting of the separatrices, that is, the distance between W^s_{po} and W^u_{po} or the angle of intersection has exponentially small upper bounds. In fact, not only the bounds are exponentially small, but the actual value of the splitting is exponentially small [3]. Hence the associated "chaoticity" or "stochasticity" domain is also exponentially small. This kind of behaviour is quite general, as we shall see, and hence we can predict if the size of the chaotic zone is relevant or not. For instance, in [8] it is proved that the motion of a satellite around an oblate primary (as it happens with artificial Earth satellites if we only take into account the J_2 term of the potential as perturbation) is non integrable. This implies the existence of chaotic zones, but in [18] it is proved that they can be neglected for all practical purposes.

We return to (11). We can consider the motion just for times of the form $2\pi\varepsilon k$, $k \in \mathbb{Z}$ (this is a kind of stroboscopic observation). Let (q_k, p_k) be the coordinates at time $2\pi\varepsilon k$. The map

$$(q_k, p_k) \xrightarrow{P} (q_{k+1}, p_{k+1}) \tag{12}$$

is independent of k and is called the map "time the period". We remark that this is a discret map from $\mathbb{S}^1 \times \mathbb{R}$ into itself. In fact it is an analytic diffeomorphism (i.e., analytic, 1 to 1, and P^{-1} is also analytic). Furthermore it is ε-close to the identity and it has two fixed points, corresponding to the two periodic orbits of period $2\pi\varepsilon$. The point corresponding to the unstable periodic orbit is hyperbolic, that is, the eigenvalues of DP at that point are different from 1 in modulus. But these eigenvalues are of the form $\lambda = 1 + a\,\varepsilon + O(\varepsilon^2),\ \lambda^{-1}$.

In general, given an equation like (3) not depending on time (i.e., autonomous) and having a periodic orbit, γ, through some point x_0, it is possible to define a map similar to P as follows: Take any hypersurface Σ (a manifold of dimension $n-1$) transversal to f at x_0 (this means that $f(x_0)$ is not contained in the tangent space to Σ at x_0, $T_{x_0}\Sigma$). The point x_0 returns to Σ under the flow, because it is on a periodic orbit. Hence nearby points must also return. Consider a point x_1 close to x_0 and the first return of x_1 to Σ close x_0. That is $\varphi(t_1;\, t_0, x_1)$ where $t_1 = \min\{t > t_0 \mid \varphi(t;\, t_0, x_0) \in \Sigma\}$. That point $\varphi(t_1;\, t_0, x_1)$ is called the Poincaré image of x_1, $P\,x_1$, and the map $x_1 \longrightarrow P\,x_1$ is called the Poincaré map. If f and Σ are analytic so is P. If the system (3) is Hamiltonian with m degrees of freedom (and hence $n = 2m$), we can restrict P to $\Sigma \cap H^{-1}(H(x_0))$ and we obtain that P is a symplectic map on a manifold of dimension $2m-2$. If $m = 2$ this is, locally, a map of the plane into itself and it is area preserving.

Consider now an analytic area preserving map ε-close to the identity having the origin as fixed point:

$$P_\varepsilon : \begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} x + \varepsilon\, g_1(x,y) + O(\varepsilon^2) \\ y + \varepsilon\, g_2(x,y) + O(\varepsilon^2) \end{pmatrix} \quad , \quad P_\varepsilon \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{13}$$

$$Spec\ DP_\varepsilon \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \{\lambda, \lambda^{-1}\}, \quad \lambda = \lambda(\varepsilon) = 1 + a\,\varepsilon + O(\varepsilon^2), \quad a > 0\,.$$

From (13) it follows that one can define a limit flow on the plane by

$$\dot{x} = g_1(x,y), \quad \dot{y} = g_2(x,y). \tag{14}$$

It turns out that (14) is Hamiltonian. Assume that (14) has a separatrix $\gamma(t)$, $t \in \mathbb{R}$, that is $\gamma(t) \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for $t \to \pm\infty$. Consider γ as a function of complex time. Let δ the minimum distance from the real axis to the singularities of γ (these singularities always exist and δ is always positive).

Theorem 2 [4]. *Under the hypothesis above for any ε such that $|\varepsilon| < \varepsilon_0$, $\varepsilon_0 > 0$ and for any $\eta \in (0, \delta)$ the splitting of the separatrices has the upper bound*

$$M(\eta)\ \exp\left(-\frac{2\pi(\delta - \eta)}{\log\ \lambda(\varepsilon)}\right).$$

For a fixed η this bound is uniform for all $|\varepsilon| < \varepsilon_0$.

The proof of theorem 2 has several steps: local, global and analytic estimates. Essentially one proves first the existence of a Birkhoff normal form with domain of convergence uniform with respect to ε. In that domain one can define an energy function in a natural way (just the product of the normal coordinates). The separatrix of the limit flow is bounded for any t satisfying $|Im\,(t)| \leq \delta-\eta$. The stable and unstable manifolds of the hyperbolic point under P_ε are shown to be close to the separatrix of the limit flow on that domain by using suitable parameterizations of W^s, W^u by means of a complex parameter t such that along these manifolds the shift $t \to t + \log \lambda(\varepsilon)$ is conjugated to P_ε. The energy is transported along W^u by analytic continuation and then is it evaluated on W^s. One obtains a $\log \lambda(\varepsilon)$-periodic function and from this the result follows.

For general diffeomorphisms (either conservative or not) one can use a suspension technique to derive a flow such that a suitable Poincaré map of it coincides with the given diffeomorphism. In general the flow is not analytic, the dependence of the vector field on t being just $\mathcal{C}^r$ for some positive r, or eventually $\mathcal{C}^\infty$, by an Hermite interpolation procedure. But this is enough to apply the results on averaging if the diffeomorphism is close to the identity. I refer to [2] for an application to the family of Bogdanov-Takens diffeomorphisms. As a Corollary, the dynamics of the $B-T$ diffeomorphisms is identical to the one of the $B-T$ vector fields except on an exponentially narrow domain on the parameters space.

4 The quasiperiodic Floquet theory and the existence of quasiperiodic solutions

Consider the linear differential equation

$$\frac{dx}{dt} = A\,(t)\;x, \tag{15}$$

where $A\,(t)$ depends on time periodically. The classical Floquet theory assures that a linear time periodic change of coordinates, $y = P\,(t)\,x$, reduces the system to $\dot{y} = By$, where B has constant coefficients. We say, simply, that in the periodic case (15) has the reducibility property. A natural question is to ask whether (15) has the reducibility property if $A\,(t)$ is quasiperiodic.

The answer is negative, and there are examples displaying non reducibility. We can be less ambitious and formulate the same question in the perturbative case

$$\frac{dx}{dt} = (A + \varepsilon\,Q\,(t))\,x, \tag{16}$$

where A has constant coefficients, Q is quasiperiodic analytic on a strip of width ρ and ε is small enough. In that case the answer is positive if A is hyperbolic (see [1]), that is, all the eigenvalues of A have non zero real part. In the general case, that is A arbitrary, the best available result is

Theorem 3 [9]. *Assume that in (16) Q is quasiperiodic analytic on a strip of width ρ with basic frequencies $\omega_1, \ldots, \omega_m$. Let $\bar{Q} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} Q(t)\,dt$ be the average of Q, and consider the matrix $\bar{A}(\varepsilon) = A+\varepsilon\bar{Q}$. Let $\lambda_j(\varepsilon)$ be the eigenvalues of $\bar{A}(\varepsilon)$ for $j = 1, \ldots, n$. Assume $\frac{d}{d\varepsilon}\,\lambda_j(\varepsilon) \neq 0$, $\frac{d}{d\varepsilon}\,(\lambda_j(\varepsilon) - \lambda_k(\varepsilon)) \neq 0$ for all $j, k \in \{1, \ldots, n\}$, $k \neq j$. Finally assume that the eigenvalues of A and the basic frequencies satisfy the diophantine conditions $|\lambda_j(0) + i\sum_{r=1}^{m} k_r\,\omega_r| > \frac{c}{(1+|k|)^\tau}$, $|\lambda_j(0) - \lambda_k(0) + i\sum_{r=1}^{m} \omega_r\,k_r| > \frac{c}{(1+|k|)^\tau}$*

for all $k \in \mathbb{Z}^m$ and some positive constants c, τ. Then there is some $\varepsilon_0 > 0$ and a set $\mathcal{E} \subset [-\varepsilon_0, \varepsilon_0]$ such that for all $\varepsilon \in \mathcal{E}$ the equation (16) has the reducibility property. Furthermore the set $\mathcal{E}$ is cantorian and for all $\varepsilon_1 \in (0, \varepsilon_0]$ the measure of $[-\varepsilon_1, \varepsilon_1] \backslash \mathcal{E}$ is exponentially small with respect to ε_1.

The proof of Theorem 3 strongly relies on estimates obtained from the use of complex time. For all $\varepsilon \in \mathcal{E}$ there is an infinite sequence of transformations converging on a domain of width less than ρ (for instance $\rho/2$ or, in general, ρ/a for any given fixed $a > 1$). After s steps of the transformation sequence the current equation is similar to (16) but with A replaced by some $A_s(\varepsilon), Q(t)$ replaced by $Q_s(t, \varepsilon)$ and ε replaced by ε^{2^s}. The diophantine conditions prevent from exact resonance at $\varepsilon = 0$ between the eigenvalues of A (and their differences) and the basic frequencies. The conditions on the variation of λ_j with ε prevent from being locked at resonance and can be weakened without too much effort [11].

A more general problem is the following. Consider a fixed point of a system of autonomous equations

$$\frac{dx}{dt} = f(x) = Ax + h(x) \ , \ \ h(x) = O\left(\|x\|^2 \right). \tag{17}$$

Then assume that we perturb (17) to

$$\frac{dx}{dt} = (A + \varepsilon\, Q(t, \varepsilon))x + \varepsilon\, g(t, \varepsilon) + h\,(x, t, \varepsilon), \tag{18}$$

where the dependence with respect to x in h is analytic, $h\,(x, t, \varepsilon) = O\left(\|x\|^2 \right)$ uniformly with respect to t and ε, the dependence with respect to t of $Q,\ g$ and h is quasiperiodic analytic on a strip of width ρ and, finally, the dependence of $Q,\ g$ and h with respect to ε is Lipschitz. Then

Theorem 4 [10]. *Assume that the eigenvalues of A and the basic frequencies appearing in $Q,\ g$ and h satisfy the same diophantine conditions of Theorem 3 and that the eigenvalues $\lambda_j(\varepsilon)$ of $\bar{A}(\varepsilon)$, defined as in Theorem 3 satisfy upper and lower Lipschitz conditions, that is $c_1|\varepsilon_1 - \varepsilon_2| < |\lambda_j(\varepsilon_1) - \lambda_j(\varepsilon_2)| < c_2|\varepsilon_1 - \varepsilon_2|$ for some positive constants $0 < c_1 < c_2$ and similar conditions for all the differences $\lambda_j(\varepsilon) - \lambda_k(\varepsilon),\ j \neq k$. Then there exist ε_0 and a set $\mathcal{E}$ as in Theorem 3 such that $\forall\, \varepsilon \in \mathcal{E}$ the system (18) has a quasiperiodic solution.*

The Theorem 4 has been motivated by the motion of a spacecraft in the vicinity of libration points in the solar system. When we consider two primaries (for instance Earth and Moon, or Sun and Earth, Sun and Jupiter, etc.) in circular orbits, the motion of a small mass (a spacecraft, an asteroid, etc.) near one of the libration points of Lagrange or Euler type, is described by the restricted three body model, which has the form (17). Real motion is more involved, and for moderate time spans it is well described by (18), where the influence of the non circular motion of the primaries and the effect of the remaining bodies of the solar system can be assumed to be quasiperiodic. The natural question is to ask whether (18) has a quasiperiodic solution replacing the fixed point of (17). The answer is provided by Theorem 4.

The proof is now rather involved. It has to be carried out by using an infinite sequence of transformations, and each transformation, in turn, is split in two different steps. Allowing for Lipschitz dependence on ε creates some technical difficulties, but they can not be avoided by accepting better dependence on ε, because one has to work on a set of values of $\varepsilon, \mathcal{E}$, which is totally disconnected.

As before, if A is hyperbolic then the set $\mathcal{E}$ is the full set $[-\varepsilon_0, \varepsilon_0]$ and furthermore the quasiperiodic orbit lies on an invariant torus which is hyperbolic [19]. If the system is slow or the quasiperiodic time dependence is fast (see section 2) then the stable and unstable invariant manifolds of the torus (the so called whiskers of the torus) have an exponentially small splitting (see section 3).

5 Integrability of Hamiltonian systems

Consider an autonomous analytical Hamiltonian system with n degrees of freedom defined on an open set $\Omega \subset \mathbb{R}^{2n}$,

$$H = H(q, p), \tag{19}$$

the Hamilton equations being

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{q} = -\frac{\partial H}{\partial q}. \tag{20}$$

Of course, H itself is a first integral of (20). We will say that the Hamiltonian H is integrable (in the sense of Liouville-Arnold) if there are n first integrals $F_1 = H,\ F_2, \ldots,\ F_n$ of (20), functionally independent (except, perhaps, on a set of zero measure) and in involution, that is, the Poison brackets $[F_i, F_j] = \sum_{k=1}^{n} \frac{\partial F_i}{\partial q_k}\frac{\partial F_j}{\partial p_k} - \frac{\partial F_i}{\partial p_k}\frac{\partial F_j}{\partial q_k}$ are all zero.

Poincaré noticed that the existence of additional first integrals implies conditions on the behaviour of the solutions of the variational equations. This fact was used by Ziglin [21] to derive necessary conditions for integrability.

Let Γ be an integral curve, that is, a solution of (20), $q(t),\ p(t)$. Now we consider t as a complex variable. The analyticity in (19) implies that H is also analytical in some complex neighbourhood of Ω. By analytic continuation of $q(t), p(t)$ we have that Γ is defined on some domain of $\mathbb{C}$ (eventually in all $\mathbb{C}$). This defines a Riemann surface, $\mathcal{R}$. Along $\mathcal{R}$ we consider the normal variational equations. We recall that the variational equations associated to (20) are

$$\frac{d}{dt}A(t) = \left.\begin{pmatrix} \dfrac{\partial^2 H}{\partial p\,\partial q} & \dfrac{\partial^2 H}{\partial p^2} \\ -\dfrac{\partial^2 H}{\partial q^2} & -\dfrac{\partial^2 H}{\partial p\,\partial q} \end{pmatrix}\right|_{(q(t),p(t))} A(t)\,, \tag{21}$$

with $A(t_0) = Id$ at some given value of t. The system (21) consists of $(2n)^2$ equations, linear and with time dependent coefficients. From (21) we pass to normal (and reduced) variational equations (NVE) by restricting to the normal directions to the orbit and by reducing to the level of energy of Γ. The solutions are matrices of dimension $(2n-2) \times (2n-2)$.

Now consider any complex path on $\mathbb{C}$ for which $q(t), p(t)$ is defined. Assume this gives a closed path, γ, on $\mathcal{R}$. We require, of course, that the path goes through regular points of $\mathcal{R}$, avoiding the possible singularities. Starting at a point $\Gamma(t_0)$ and returning to it along γ we find some matrix, M_γ, as solution of the NVE. We denote M_γ as the monodromy matrix along γ. We can take different paths, $\gamma_1, \gamma_2, \ldots, \gamma_k$ on $\mathcal{R}$, obtaining different (eventually) matrices $M_{\gamma_1}, \ldots, M_{\gamma_k}$. This set of matrices forms a group, the monodromy group of the NVE along $\mathcal{R}, \mathcal{M}$. It is a finitely generated subgroup of the symplectic group $Sp(n-1, \mathbb{C})$, and it gives a representation of the homotopy group of $\mathcal{R}$ in $Sp(n-1, \mathbb{C})$.

An element $g \in Sp(n-1, \mathbb{C})$ is called non-resonant if the eigenvalues of g : $\lambda_1, \ldots, \lambda_{n-1}, \lambda_1^{-1} \ldots, \lambda_{n-1}^{-1}$ do not satisfy any relation of the form $\prod_{j=1}^{n-1} \lambda_j^{k_j} = 1,\ k_j \in \mathbb{Z}$ for $j = 1, \ldots, n-1$, unless all the k_j are zero.

Theorem 5 [21]. *Suppose that the monodromy group $\mathcal{M}$ contains a non-resonant matrix, g_0. If $\mathcal{H}$ has $n-1$ first integrals in addition to $\mathcal{H}$, functionally independent and independent of $\mathcal{H}$ (but may be not in involution) and they are meromorphic in a connected neighbourhood of Γ, then any other matrix $g \in \mathcal{M}$ must send eigendirections of g_0 into eigendirections of g_0.*

From now on we shall restrict ourselves to the case of $n = 2$ degrees of freedom. See [12] for the general case. The NVE are now a system of equations for 2×2 matrices. All the columns of these matrices are solutions of a linear system of 2 differential equations or, equivalently, of a single second order linear equation. To state the results we need some additional preliminaries. Let

$$y^{(m)} + p_1\, y^{(m-1)} + \cdots + p_{m-1}\, y' + p_m\, y = 0 \tag{22}$$

an order m linear equation, where the functions $p_1, \ldots p_m$ depend on t, $y' = \frac{dy}{dt}$ and $y^{(k)} = \frac{d^k y}{dt^k}$. Assume that the functions $p_1, \ldots, , p_m$ belong to some field of functions, K. Let $y_1(t), \ldots, y_m(t)$ a fundamental system of solutions of (22), so that any solution of (22) can be written as $\sum_{j=1}^{m} \alpha_j\, y_j(t)$ with constant coefficients α_j. The solutions $y_j(t),\ j = 1, \ldots, m$ can belong to K or not. If some of them do not belong to K we can define a new field of functions $P = K\langle y_1, \ldots, y_m \rangle$ denoted as the Picard-Vessiot extension of K. Then we can consider the Galois group G of P over K, that is, the group of all (differential) automorphisms of P which leave K invariant. It turns out that G is an algebraic closed subgroup in $GL(m, \mathbb{C})$ (under the Zariski topology).

Theorem 6 [12]. *Assume that the hypotheses of Ziglin's theorem for two degrees of freedom hold and that the NVE along Γ have coefficients over a field K of meromorphic functions over a compact Riemann surface and they are of Fuchsian type. Then they are Picard-Vessiot integrable, that is, the related Galois group, G, has a solvable identity component, G_0. Furthermore there are only two possible cases:*

a) P is obtained by adjunction of exponentials of integrals of functions of K.

b) There is a chain of extensions $K \subset L \subset P$, L of second degree over K and P is obtained by exponentiation of integrals of functions of L.

I refer to [12] for additional information on the possible structures of the monodromy group $\mathcal{M}$ if H is integrable. These abstract results have been applied in [13] to some families of examples. Assume, for instance that a two degrees of freedom Hamiltonian has NVE of Lamé type:

$$\frac{d^2\xi}{dt^2} = (A\,\mathfrak{P}(t) + B)\,\xi\,, \tag{23}$$

where $\mathfrak{P}$ denotes the Weierstrass function and A and B are (complex) parameters. $\mathfrak{P}$ is a solution of $\dot{z}^2 = 4\,z^3 - g_2 z - g_3$, where g_2 and g_3 are the invariants of $\mathfrak{P}$ and it is assumed that the discriminant $\Delta = 27\,g_3^2 - g_2^3$ is nonzero. In (23) we set $A = n\,(n+1)$ (n being a complex parameter in general). In [13] the possible classical Hamiltonians $H = \frac{1}{2}\,(p_1^2 + p_2^2) + V\,(q_1, q_2)$ giving rise to NVE as (23) are given and the integrability is studied. A very simple family of Hamiltonians is the so called homogeneous Hénon-Heiles case, with

$$H\,(q,p) = \frac{1}{2}\,(p_1^2 + p_2^2) + \frac{e}{3}\,q_1^3 + q_1\,q_2^2\,. \tag{24}$$

In (24) e is a parameter. It is shown that only $e = 1, 6, 16$ give integrable Hamiltonians.

A natural question arises: What is the dynamical consequence of the lack of integrability for any e different from $1, 6, 16$? For real negative values of e the answer is quite simple: the Hamiltonian (24) has hyperbolic periodic orbits whose stable and unstable manifolds intersect transversally. This leads to non predictable dynamics. For $e > 0$ the situation is not so clear. Indeed, one checks immediately $\ddot{q}_1 = -eq_1^2 - q_2^2$. Due to the negative character of the q_1 component of the acceleration, there is no possible recurrence in the real phase space. Any real motion has q_1 going to $-\infty$ for both $t \to \pm\infty$ (unless we remain for ever at the origin).

A similar behaviour is found in the n-body problem. If we consider the Newtonian problem, it is well known that for any $n > 2$ the problem is non integrable. However, if we consider the Coulombian case, that is, the motion of n bodies having electric charges of the same sign and, hence, repelling the ones to the others, all bodies escape to infinity for $t \to \pm\infty$. No recurrence is possible and, in fact the limit velocities are seen to be independent integrals (taking out, of course, the center of mass integral), analytic and in involution. But both the Newtonian and the Coulombian problems are the same one! Just change t by $i\,t$ to go from the one to the other.

We return to the homogeneous Hénon-Heiles problem. Starting at a negative value of e we have periodic hyperbolic orbits. Then it is possible to continue e to complex values. Of course, both the coordinates and the period of the corresponding periodic orbits are complex. In this way we can reach any positive value of e. Periodic hyperbolic orbits are found (numerically) for values of e different from $1, 6, 16$, with (complex) stable and unstable manifolds and transversal intersection. It is important to remark that the continuation defines the initial conditions of the periodic orbit (in a given Poincaré and given level of H) which are analytic if the eigenvalues differ from 1. When for some $e = e^*$ the eigenvalues are equal to 1, a branching point is found, generically. Using a normal form argument the initial conditions have a branching point of order 2 at e^*, and the eigenvalues a branching point of order 4. For (24), due to some special symmetries, there are also points e^* with $\lambda = 1$ having non generic character. For e^* close to 1.51602386 a generic point occurs.

An heuristic (non rigorous) method uses the Painlevé Criterion. I refer to [17] for the description and examples.

6 Stability loss delay

Stability loss delay is an interesting, important and so far not completely clear phenomenon. Its essence is as follows [15]. Consider a system of ordinary differential equations depending on a slowly varying parameter

$$\frac{dx}{dt} = f\left(x, \tau, \varepsilon\right), \quad \frac{d\tau}{dt} = \varepsilon. \tag{25}$$

Assume (25) has a fixed point or a periodic solution for any fixed value of τ. Suppose also that for $\tau = 0$ the system passes through a bifurcation: the fixed point (or the periodic orbit) loses stability remaining non degenerate, that is, none of the eigenvalues is equal to 0 (for the periodic orbit none of the eigenvalues of the Poincaré map is 1). In the case of a fixed point a pair of conjugate eigenvalues crosses the imaginary axis from left to right (Hopf bifurcation). In the case of a periodic solution, either a pair of conjugate eigenvalues of the Poincaré map leaves the unit circle not passing through 1 (Hopf bifurcation) or one of the eigenvalues goes away from the unit circle through the point -1 (flip or period doubling bifurcation). Note that the fixed point (respectively the periodic orbit) depends on τ if τ is kept fixed.

Assume (25) is analytic. Then a delay in the stability loss is observed: the points of the phase space close to the equilibrium (the periodic orbit) of the "frozen system" (that is, the system with τ fixed) long before $\tau = 0$, remain close to the equilibrium point (or the periodic orbit), when it is unstable, until τ is $O(1)$, for any value of ε even arbitrary small. Summarizing, the stability loss is observed for some time of the order of $1/\varepsilon$.

For definiteness let us consider the Hopf bifurcation of periodic orbits. We consider τ as a complex parameter (τ is a "slow time", because $\tau = \varepsilon\, t$). At $\tau = 0$ two complex conjugate eigenvalues of the Poincaré map related to the periodic orbit leave the unit circle through points different from 1. Assume that all the other eigenvalues (if any) are stable and hyperbolic. Let λ one of these eigenvalues which cross the unit circle. Consider the function

$$\phi(\tau) = Re \int_0^\tau \frac{\log\, \lambda(s)}{T(s)}\, d\, s\,, \tag{26}$$

where $T(s)$ is the period of the periodic orbit of parameter s and $\lambda(s)$ the eigenvalue obtained from the chosen one at $s = 0$. The set of points with $\phi(\tau) = ctant$ is called isocontracting line. Near $\tau = 0$ the isocontracting lines are curves intersecting $Im\, \tau = 0$ in two points $\tau_{in} < 0$ and $\tau_{out} > 0$. Let $\sigma > 0$ a small fixed quantity. Then assume that for $\tau = \tau_{in}$ a point in the phase space is at a distance σ of the periodic orbit corresponding to the frozen value $\tau = \tau_{in}$ of the parameter. When t increases and, hence, τ increases, after passing the bifurcation value $\tau = 0$ the point obtained by evolution of the initial one remains rather close to the frozen periodic orbit of the current τ value, despite the orbit is unstable, until it reaches again a distance σ. When ε goes to zero this happens for a value which tends to τ_{out}.

This pattern subsists when τ goes away of $\tau = 0$ if all the remaining eigenvalues are still stable hyperbolic, until $\lambda(\tau)$ reaches, eventually, the value $\lambda(\tau) = 1$. At that point the lines $\phi(\tau) = ctant$ have a singularity and, generically, they form an angle of $\frac{2\pi}{3}$. I refer to [16] for the full statements in the general case and for the technical details.

7 Complex homoclinic points

A related topic appears when we skip the time dependence and we look just at the diffeomorphism induced on a Poincaré section by the flow of an autonomous equation. Points where the stable and the unstable invariant manifolds of an hyperbolic fixed point intersect transversally are called transversal homoclinic points. Interesting phenomena appear when a parameter is changed and a tangency is established between a stable and an unstable manifold. The simplest situation appears in 2D diffeomorphisms (either conservative or not). Generically the tangency is quadratic. Then there are different possibilities, according to the global position of W^u with respect to W^s, to the signs and values of the eigenvalues, etc. (see [20]).

Typically, after the tangency a horseshoe has been created. It contains hyperbolic and hyperbolic with reflection periodic points. If we consider a given period, m, for some value of the parameter before the tangency there are not periodic points of period m. After the creation of the horseshoe there are two of them. The generic situation is as follows: For same value of the parameter, say, μ_m^b a saddle-node bifurcation is produced. When moving μ towards the tangency the saddle subsists as a saddle and the node turns to a focus, then to a node with negative eigenvalues and for some new bifurcation value, μ_m^f, it has a flip or period doubling bifurcation and it gives a saddle with reflection. Of course, in the conservative case instead of a node or a focus we have elliptic periodic points.

In [20] the local basins of attraction of the node/focus is obtained in the dissipative case, and the dynamical consequences are studied.

The same pattern repeats itself for all periods m larger then some minimal value, m_0. The sequence $\{\mu_m^b\}$ accumulates to the value of the parameter at the tangency, μ_t.

All these bifurcations become more clear if we study analytic diffeomorphisms and we consider the dynamics in the complex phase space. The transversal homoclinic points do not disappear at the homoclinic tangency: they just move to the complex phase space. They remain transversal in that phase space and, hence, all the rich dynamics associated to them. The same is true for the periodic orbits. The "classical" Hénon map is a good example of this. For a given number m the number of points of period m (or a divisor of m) is always 2^m. But only part of them "are seen" on the real phase space.

An example of a study of the dynamics in $\mathbb{C} \times \mathbb{C}$ can be found in [5], where the standard and semistandard maps are analyzed. The standard map can be put in the form

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \begin{pmatrix} x + \bar{y} \\ y + \varepsilon \sin x \end{pmatrix}. \tag{27}$$

It is immediately checked that it has an elliptic fixed point at $(\pi, 0)$ and an hyperbolic one at (0,0). The invariant manifolds of the origin under (27) are diffeomorphic to complex lines (of real dimension 2) when (27) is considered in $\mathbb{C} \times \mathbb{C}$. The complex homoclinic points are the tips of the leaves of some fern-like structure on the invariant manifolds. In some sense most of the dynamics is present for any non zero value of ε, but only a part of it lies on the real domain.

Notice, however, that, in principle, a canonical map in $\mathbb{C} \times \mathbb{C}$ can be considered as a canonical map in $\mathbb{R}^4$. But it has many more constraints than a generic canonical map in $\mathbb{R}^4$. In particular it preserves two independent real symplectic structures (the real and the imaginary part of the complex $\omega = dx \wedge dy$), at all the fixed or periodic points all the eigenvalues become double, etc. This is similar to the difference between an analytic endomorphism in $\mathbb{C}$ and a real analytic one in $\mathbb{R}^2$.

8 Final comments

The goal of this lecture is to stress on the relevance of the use of complex time and complex phase space to study real dynamics. Not only the global pictures of bifurcations become more clear, but also many estimates, difficult to obtain, are relatively easier using complex variables.

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